On Factor Complexity of Power-Free Words

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Outline

Introduction

Two central topics in combinatorics on words:

- Power avoidance in finite and infinite words
- Subword/factor complexity of infinite words

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General Questions

1. Which subword complexities are available under a given power avoidance restriction?
2. What powers can be avoided by words from a given complexity class?

Aim of this talk: present some interesting results on the first question and motivate the further study

Notation and Definitions

- Array notation for words: $w = w[1..n]$, $n = |w|$
  - $\mathbf{w} = \mathbf{w}[1..\infty]$ or $\mathbf{w} = \mathbf{w}[0..\infty]$ whichever is more convenient
- Subword/factor $w[i..j]$, prefix $w[1..i]$, suffix $w[j..n]$, $\mathbf{w}[j..\infty]$
  - $v \prec w$: v is a factor of w
  - $L(\mathbf{w}) = \{v \mid v \prec \mathbf{w}\}$: language of $\mathbf{w}$
- Subword/factor complexity $p_{\mathbf{w}}(n) = \#\{v \mid v \prec w, |v| = n\}$
- Period p: $w[1..n{-}p] = w[p{+}1..n]$, $\mathbf{w}[1..\infty] = \mathbf{w}[p{+}1..\infty]$
  - $per(w)$: the minimum period of w
  - aperiodic infinite word: no suffix has a period
- Exponent: $exp(w) = |w|/per(w)$
- Local/critical exponent: $lexp(w) = \sup\{exp(v) : v \prec w\}$
- w avoids power $\alpha > 1$, $\alpha \in \mathbb{R}$ if $lexp(w) < \alpha$
  - a.k.a. "w is α-power-free"
  - w avoids α^+ if $lexp(w) \geq \alpha$
  - same for infinite words
  - α (or α^+) is k-avoidable if some k-ary infinite w avoids it

Something On Power-Free Languages

Threshold Theorem (Dejean's Conjecture)

Let repetition threshold $RT(k)$ be the following function:

k	2	3	4	5	$\ldots$	n	$\ldots$
$RT(k)$	2	$7/4$	$7/5$	$5/4$	$\ldots$	$n/(n-1)$	$\ldots$

Then the minimum k-avoidable power is $RT(k)^+$.

Growth of infinite power-free languages (of finite words):

- binary: polynomial for $\alpha \leq 7/3$, exponential for $\alpha \geq (7/3)^+$
- k-ary, $k > 2$: exponential (conjecture!)
  - confirmed for $kle10$ and for odd k up to 101

Alphabets:

- here we study only binary and ternary words

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Given a set $\mathcal{L}_{k,\alpha}$ of all k-ary α-power-free infinite words and the set $C_{k,\alpha}$ of their complexities, we are interested in words of

- minimum complexity
  - each other complexity in $C_{k,\alpha}$ is bigger
- minimal complexity
  - no other complexity in $C_{k,\alpha}$ is smaller
- asymptotically minimum/minimal
- of minimal asymptotic growth

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- minimal complexity
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- asymptotically minimum/minimal
- of minimal asymptotic growth
- same for maximum/maximal
- same for symmetric words
  - **w** is symmetric if $v \prec \mathbf{w}$ implies $\pi(v) \prec \mathbf{w}$ for any permutation π of the alphabet

What Are Extremal Words?

♡ Definitely, some old friends

- Thue-Morse word
- Fibonacci word
- Arshon word
- ternary Thue word
- ...

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♠ And some more or less ugly constructions as well ...

Outline

Polynomial Plateau: Thue-Morse Word

$\mathbf{w} \in \{0,1\}^{\omega}$ is α-power-free; $2^+ \leq \alpha \leq 7/3$

- 2^+-power-free (overlap-free): no factors $XXX[1]$
- $(7/3)$-power-free: no factors $XXX[1..i]$, where $i > |X|/3$

Everything is very close to the Thue-Morse word

$$\mathbf{t} = \mathbf{t}[0..\infty] = 0110100110010110\cdots,$$

which is the fixed point of the morphism $\mu : 0 \to 01, 1 \to 10$.

Proposition

Every $(7/3)$-power-free (in particular, overlap-free) binary infinite word contains all factors of the Thue-Morse word.

Corollary

Thue-Morse word $\mathbf{t}$ has the minimum complexity over all $(7/3)$-power-free (e.g., overlap-free) binary infinite words.

Polynomial Plateau: Thue-Morse Word (2)

More facts on the Thue-Morse word **t**:

- $$p_{\mathbf{t}}(n+1) = \begin{cases} 4n - 2^i, & \text{if } 2^i \le n \le 3 \cdot 2^{i-1}; \\ 2n + 2^{i+1}, & \text{if } 3 \cdot 2^{i-1} \le n \le 2^{i+1}. \end{cases}$$
  - $3n + O(1) \le p_{\mathbf{t}}(n) \le \frac{10}{3} n + O(1)$
- **t** is symmetric
- all symmetric (7/3)-power-free binary infinite words share the same language of factors $L(\mathbf{t})$
  - minimum complexity = maximum complexity !
- the set of words with language $L(\mathbf{t})$ has the cardinality of continuum

Thue-Morse vs twisted Thue-Morse

Another definition of the Thue-Morse word $\mathbf{t} = \mathbf{t}[0..\infty]$:

- $\mathbf{t}[i]$ is the number of 1's (mod 2) in the binary expansion of i

Twisted Thue-Morse word $\mathbf{t}' = \mathbf{t}'[0..\infty]$ is defined by

- $\mathbf{t}'[i]$ is the number of 0's (mod 2) in the binary expansion of i
  - no leading zeroes: the expansion of 0 is the empty word!

$$\mathbf{t}' = 0010011010010110011010011001011010 01\cdots = 00\mu(1)\mu^2(0)\cdots\mu^{2n}(0)\mu^{2n+1}(1)\cdots$$

- is overlap-free
- is not symmetric (e.g., no 11011)
- is very similar to, and very dissimilar with, the word $\mathbf{t}$

Thue-Morse vs twisted Thue-Morse (2)

Similarity: rank sequences

- we call $\mu^k(0), \mu^k(1)$ k-blocks; 0,1 are 0-blocks
- given $\mathbf{w} \in \{0,1\}^\omega$, position i has rank $r_{\mathbf{w}}(i) = k$ if
  - $\mathbf{w}[i..\infty]$ is a product of k-blocks (and of j-blocks for $j < k$)
  - $\mathbf{w}[i..\infty]$ is not a product of $k{+}1$-blocks

⋆ a position can have an infinite rank

⋆ sequence of ranks determines **w** up to renaming letters

$$\begin{array}{rcccccccccccccccccc} \mathbf{t} & = & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & \cdots \\ r_{\mathbf{t}}(i) & = & \infty & 0 & 1 & 0 & 2 & 0 & 1 & 0 & 3 & 0 & 1 & 0 & 2 & 0 & 1 & 0 & 4 & \cdots \end{array}$$

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\end{array}
$$

Dissimilarity: Hamming distance

$\liminf_{i\to\infty} H(\mathbf{t}[1..i], \mathbf{t}'[1..i])/i = 1/3$ (minimum possible)

$\limsup_{i\to\infty} H(\mathbf{t}[1..i], \mathbf{t}'[1..i])/i = 2/3$ (maximum possible)

Polynomial Plateau: Maximum Complexity

Theorem

The twisted Thue-Morse word $\mathbf{t}'$ has maximum subword complexity over all overlap-free infinite binary words, and is the unique word with this property, up to renaming letters.

★ $\frac{13}{4}n + O(1) \le p_{\mathbf{t}'}(n) \le \frac{7}{2}n + O(1)$

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Theorem

There is no $(7/3)$-power-free binary infinite word of maximum complexity.

- Open: is there a $(7/3)$-power-free binary infinite word of asymptotically maximum complexity? What is the maximum asymptotic growth?
  - $p_{\mathbf{w}}(n) < \frac{6}{5}p_{\mathbf{t}}(n)$ for every $n \ge 0$, $\mathbf{w} \in \mathcal{L}_{2,\frac{7}{3}}$

Outline

Binary words: Above 7/3

Theorem

There is a (morphic) $(\frac{7}{3})^+$-power-free binary infinite word the complexity of which is asymptotically incomparable with $p_{\mathbf{t}}(n)$.

- Open: what is the minimum α such that the Thue-Morse word does not have minimal complexity over all α-power-free binary infinite words? over all symmetric α-power-free binary infinite words?

Binary words: Climbing higher

Theorem

There is a (morphic) $(\frac{5}{2})^+$-power-free binary infinite word of complexity $2n$. All $(\frac{5}{2})$-power-free binary infinite words have bigger complexity.

- Open: is the complexity $2n$ minimum over $(\frac{5}{2})^+$-power-free binary words? cube-free binary words? 3^+-power-free binary words?

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★ For $\alpha = 2+\phi$, where $\phi \approx 1.618$ is the golden ratio, there is an α-power-free binary infinite word (Fibonacci word)

★ $2+\phi$ is the minimum power avoided by Sturmian words and moreover by words of complexity $n + O(1)$

Ternary words: Square-free

The ternary Thue word:

$$\mathbf{T} = \mathbf{T}[1..\infty] = 012021012102012021020121012021012102\cdots$$

- Fixed point of the morphism $\theta : 0 \to 012, 1 \to 02, 2 \to 1$
- Square-free: $lexp(\mathbf{T}) = 2$ is not reached
- for any $i \geq 1$, $\mathbf{T}[i]$ is the number of zeroes between the i'th and $(i{+}1)$'th occurrences of 1 in $\mathbf{t} = 01101001\cdots$
- $\mathbf{T}[i] = \begin{cases} 0, & \text{if } \mathbf{t}[i{-}1..i] = 01; \\ 1, & \text{if } \mathbf{t}[i{-}1] = \mathbf{t}[i]; \\ 2, & \text{if } \mathbf{t}[i{-}1..i] = 10. \end{cases}$
- $p_{\mathbf{T}}(n) = p_{\mathbf{t}}(n{+}1)$ for all $n \geq 2$ (a bijection between factors)

Ternary words: Square-free (2)

Conjecture

The ternary Thue word **T** has the minimum subword complexity over all square-free ternary infinite words.

Theorem

1. $p_{\mathbf{T}}(n)$ is a minimal element of $\mathcal{C}_{3,2}$
2. If a word **U** has minimum subword complexity over all square-free ternary infinite words, then not only $p_{\mathbf{U}}(n) = p_{\mathbf{T}}(n)$, but $L(\mathbf{U}) = \pi(L(\mathbf{T}))$ for a bijective coding π.

Ternary words: Square-free (3)

1-2-bonacci word $\mathbf{F}_{12}$:

- Take the Fibonacci word $\mathbf{f}_{12} \in \{2, 1\}^\omega$
  - fixed point of the morphism $\varphi : 2 \to 21, 1 \to 2$
- Build $\mathbf{F}_{12} \in \{0, 1, 2\}^\omega$ inductively as follows:
  - $\mathbf{F}_{12}[1..2] = 01$; $\mathbf{F}_{12}[i] \neq \mathbf{F}_{12}[i-1]$ for all i
  - let $\mathbf{F}_{12}[1..i-1]$ be defined; next letter of $\mathbf{f}_{12}$ is
    1: let $\mathbf{F}_{12}[i] = \mathbf{F}_{12}[i-2]$, $\mathbf{F}_{12}[i+1] \neq \mathbf{F}_{12}[i-1]$
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$$\mathbf{f}_{12} =$$
$$\mathbf{F}_{12} = 01$$

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$$\mathbf{f}_{12} = \quad\quad 2$$
$$\mathbf{F}_{12} = 01\,021$$

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$$
\begin{aligned}
\mathbf{f}_{12} &= \quad\ \ 2 \quad 1 \\
\mathbf{F}_{12} &= 01\ 021\ 20
\end{aligned}
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Ternary words: Square-free (3)

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- Detailed explanations (walks in the $K_{3,3}$ graph) are skipped
- The words $\mathbf{F}_{13}$ and $\mathbf{F}_{23}$ are also useful

Ternary words: Square-free (4)

The 1-2-bonacci word

- is square-free: $lexp(\mathbf{F}_{12}) = 11/6$, reached
- is symmetric
- avoids all 5-letter factors of the form *abcab*
- has a strong extremal property related to square-free partial words

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The 1-2-bonacci word $\mathbf{F}_{12}$ has the minimum subword complexity over all symmetric square-free ternary infinite words. This complexity equals $6n - 6$ for all $n \geq 2$.

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Theorem

The 1-2-bonacci word $\mathbf{F}_{12}$ has the minimum subword complexity over all symmetric square-free ternary infinite words. This complexity equals $6n - 6$ for all $n \geq 2$.

- Open: what is the minimum power avoided by symmetric ternary infinite word of complexity $6n + O(1)$?
  - the 1-3-bonacci word $\mathbf{F}_{13}$ avoids $1 + \phi/2 \approx 1.809$

Ternary words: $(7/4)^+$-power-free

- The Arshon word (which is a fixed point of a combination of two morphisms)
  - is $(\frac{7}{4})^+$-power-free
  - is symmetric
  - has complexity $12n + O(1)$
- Open: does Arshon word have minimal/minimum complexity over all symmetric $(\frac{7}{4})^+$-power-free ternary infinite words? What about the general (non-symmetric) case?

Ternary words: above square-free

- Open: almost everything...
- The only result: there exists a pure morphic $(5/2)$-power-free word of complexity $2n+1$

Words of big (exponential) complexity

Theorem

1. There is a $(\frac{7}{3})^+$-power-free binary infinite word having exponential subword complexity.
2. There is a $(\frac{7}{4})^+$-power-free ternary infinite word having exponential subword complexity.

- Open: what is the maximum growth rate of such a complexity function? Can it be equal to the growth rate of the corresponding $\mathcal{L}_{k,\alpha}$ language?
  - Growth rate of an exponentially growing function $f(n)$ is $\limsup_{n\to\infty}(f(n))^{1/n}$
  - For a language, this is the growth rate of its growth function
  - For $\mathcal{L}_{k,\alpha}$, $\alpha \geq 2$, growth rates are known with high precision

Restivo-Salemi Property

Restivo-Salemi problem (1985):

- given two square-free ternary words u and v, how to decide whether there exists a ternary word w such that uwv is square-free? how to find such a w if it exists?

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Given a language L, we say that

- $w \in L$ is two-sided extendable in L if for every $n \geq 0$ there exist u_n, v_n such that $|u_n|, |v_n| \geq n$ and $u_n w v_n \in L$
- L has Restivo-Salemi property if for any words u, v that are two-sided extendable in L, there is w such that $uwv \in L$

Conjecture (S., 2009)

All infinite languages $\mathcal{L}_{k,\alpha}$ have the Restivo-Salemi property.

- Confirmed only for small binary languages

Words of very big complexity

Theorem

A power-free language $\mathcal{L}_{k,\alpha}$ has the Restivo-Salemi property if and only if all words from $ext(\mathcal{L}_{k,\alpha})$ are factors of some α-power-free recurrent k-ary infinite word.

★ Let $ext(L)$ be the set of all words that are two-sided extendable in L. Then $ext(L)$ has the same growth rate as L (S. 2008)

Corollary

If a power-free language $\mathcal{L}_{k,\alpha}$ possesses the Restivo-Salemi property, then there is an α-power-free k-ary infinite word with subword complexity having the same growth rate as $\mathcal{L}_{k,\alpha}$.

The talk is based on the preprint
Subword complexity and power avoidance
J. Shallit, A.M. Shur - arXiv preprint arXiv:1801.05376, 2018

Thank you for your attention!