Avoidability of Binary Patterns in the Abelian Sense

Matthieu Rosenfeld

February 22, 2018

Patterns in the usual case

A **pattern** P is a word over an alphabet Δ.

A word $w \in \Sigma^*$ **realizes** the pattern $P \in \Delta^*$ iff there are $u_1, \ldots, u_{|P|} \in \Sigma^+$ such that $w = u_1 \ldots u_{|P|}$ and $\forall i, j$ $P_i = P_j \implies u_i = u_j$.

Example

$\overbrace{0}^{A}\overbrace{102}^{B}\overbrace{102}^{B}\overbrace{0}^{A}$ realizes the pattern $ABBA$.

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- $012\overbrace{01}^{A}\overbrace{01}^{A}\overbrace{1}^{B}\overbrace{1}^{B}020$ contains a realization of $AABB$,
- 01**2020**1 contains a realization of AA,
- 0001001 avoids $AABB$.

Avoidability of AA and AAA

Question

What is the smallest n such that there is an infinite word avoiding AA over n letters?

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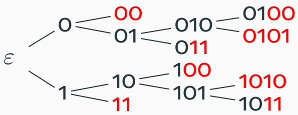

$\Rightarrow$ *AA* is not avoidable over 2 letters.

Avoidability of AA and AAA

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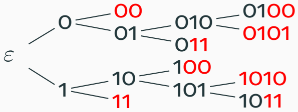
What is the smallest *n* such that there is an infinite word avoiding AA over *n* letters?



⇒ *AA* is not avoidable over 2 letters.

Theorem (Thue, 1906)

There is an infinite ternary word which avoids AA.

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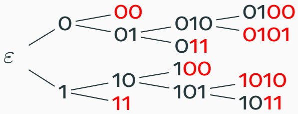

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Theorem (Thue 1906)

There is an infinite binary word (the Thue-Morse word) which avoids AAA.

Binary patterns [P. Goralcik and T. Vanicek, 1991]

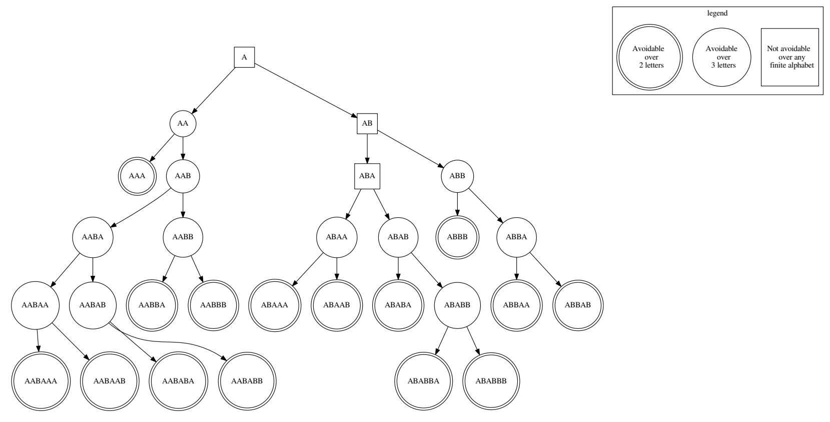

Figure 1: Classification of binary patterns by avoidability.

Every pattern that does not appear in this tree is avoidable over the binary alphabet.

Abelian equivalence and patterns

Definition

Two words u and v are **abelian equivalent**, denoted by $u \sim_a v$, if v is a permutation of u.

$aaabb \sim_a ababa$, $cabac \sim_a abcac$.

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A word $w \in \Sigma^*$ **realizes in the abelian sense** the pattern $P \in \Delta^*$ if there are $u_1, \ldots, u_{|P|} \in \Sigma^+$ such that $w = u_1 \ldots u_{|P|}$ and $\forall i, j$ $P_i = P_j \implies u_i \sim_a u_j$.

$\overbrace{201}^{A}\overbrace{210}^{A}$ realizes AA in the abelian sens.

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$\overbrace{201}^{A}\overbrace{210}^{A}$ realizes AA in the abelian sens.

002201201200 realizes ABCBCA in the abelian sense:

$$\overbrace{0}^{A}\ \overbrace{02}^{B}\ \overbrace{201}^{C}\ \overbrace{20}^{B}\ \overbrace{120}^{C}\ \overbrace{0}^{A}$$

Avoiding A^n

Question (Erdős 1957 & 1961)

Is there an infinite word over 4 letters avoiding AA in the abelian sense?

Theorem (Keränen, 1992)

The fixed points of the following 85-uniform morphism avoid AA:

$$h: \begin{cases} a \longrightarrow abcacdcbcdcadcdbdabacabadbabcbdbcbacbcdcacbabdabacadcbcdcacdbcbacbcdcacdcbdcdadbdcbca \\ b \longrightarrow bcdbdadcdadbadacabcbdbcbacbcdcacdcbdcdadbdcbcabcbdbadcdadbdacdcbdcdadbdadcadabacadcdb \\ c \longrightarrow cdacabadabacbabdbcdcacdcbdcdadbdadcadabacadcdbcdcacbadabacabdadcadabacabadbabcbdbadac \\ d \longrightarrow dabdbcbabcbdcbcacdadbdadcadabacabadbabcbdbadacdadbdcbabcbdbcabadbabcbdbcbacbcdcacbabd \end{cases}$$

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Theorem (Dekking, 1979)

The fixed points of f avoid AAA and the fixed point of h avoids $AAAA$:

$$f : \begin{cases} a & \mapsto aabc \\ b & \mapsto bbc \\ c & \mapsto acc \end{cases} \qquad h : \begin{cases} a & \mapsto abb \\ b & \mapsto aaab \end{cases}$$

Starting the classification of binary patterns

P	$AAAA$	AAA	AA
Smallest $\|\Sigma\|$ such that P is avoidable over Σ	2	3	4

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Lemma

Any realization w of $ABAB$ is also a realization of AA.

Proof.

$w = a_1 b_1 a_2 b_2$ and $a_1 \sim_a a_2$, $b_1 \sim_a b_2$

$\implies a_1 b_1 \sim_a a_2 b_2$. w is a realization of AA. $\square$

$\implies$ $ABAB$ is avoidable over 4 letters.

Starting the classification of binary patterns

Theorem (Divisibility)

Let P and O be two patterns, if

- *O does not avoid P in the abelian sense,*
- *P is abelian-avoidable over k letters*

then O is abelian-avoidable over k letters.

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Binary patterns other than A, AB and ABA are avoidable over 4 letters.

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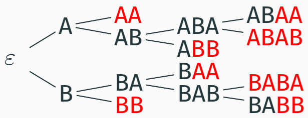
Theorem

Binary patterns other than A, AB and ABA are avoidable over 4 letters.

AA is avoidable over 4 letters and the binary patterns avoiding *AA* are:



□

Binary patterns

Theorem (2-avoidability (J. Currie, T. Visentin, 2007))

Binary patterns of length greater than 118 are abelian-avoidable over the binary alphabet.

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Binary patterns of length at least 9 are abelian-avoidable over 3 letters.

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Theorem (2-avoidability (R.))

Binary patterns of length greater than 14 are abelian-avoidable over the binary alphabet.

Over the ternary alphabet

Theorem (R.)

For any convenient morphism h and any pattern P one can decide if the fixed points of h avoid P.

(A generalization of the algorithm from Currie and Rampersad for the patterns of the form $AAA\ldots$).

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$a \mapsto aabaac,\ b \mapsto cbbbab,\ c \mapsto cbccac.$

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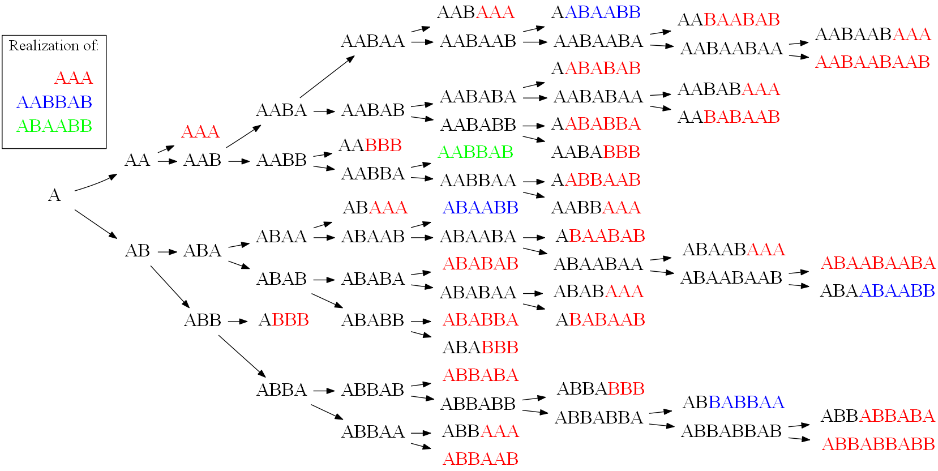
AABBAB is avoided by any fixed point of
$a \mapsto aabaac,\ b \mapsto cbbbab,\ c \mapsto cbccac.$

$\implies$ AABBAB, ABAABB and AAA are avoidable over the ternary alphabet.

$\implies$ If a pattern contains AABBAB or ABAABB or AAA it is abelian-avoidable over the ternary alphabet.

The search


Realization of:
AAA
AABBAB
ABAABB
A
AA
AB
AAA
AAB
ABA
ABB
AABA
AABB
ABAA
ABAB
ABBB
ABBA
AABAA
AABAB
AABBB
AABBA
ABAAA
ABAAB
ABABA
ABABB
ABBAB
ABBAA
AABAAA
AABAAB
AABABA
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AABBAA
ABAABB
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AABAABA
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AABABAA
AABABBA
AABABBB
AABBAAB
AABBAAA
ABAABAB
ABAABAA
ABABAAA
ABABAAB
ABBABBB
ABBABBA
AABAABAB
AABAABAA
AABABAAA
AABABAAB
ABAABAAA
ABAABAAB
ABBABBAA
ABBABBAB
AABAABAAA
AABAABAAB
ABAABAABA
ABAABAABB
ABBABBABA
ABBABBABB

Ternary alphabet

Theorem (R.)

Any binary pattern avoiding AABBAB, ABAABB and AAA belongs, up to symmetry, to: {A, AA, AB, AAB, ABA, AABA, AABB, ABAB, ABBA, AABAA, AABAB, AABBA, ABAAB, ABABA, AABAAB, AABABA, AABABB, AABBAA, ABAABA, AABAABA, AABABAA, ABBABBA, AABAABAA, ABAABAAB}.

Any other binary pattern is abelian-avoidable over 3 letters.

Over the binary alphabet

morphisms	avoided patterns
$a \mapsto aabaa$ $b \mapsto bbabb$	AABBBAAAB, ABAAABBBA, AAABABABBB, AAABABBABB,AAABABBBAB, AABBBABAAB,AABBBABABA, ABAABABBBA,ABAABBBABA, ABABAABBBA, ABBBABAAAB, AABAABBBAB,AABBBAABAB, AABBBAABAAB, AAABABBAAAB, AABBBABBBAA, ABABABBBABA, ABABBABBABA, AAABAAABBAB, AAABBABAAAB, AAABAABAABAB,AAABABAAABAB, AABAAABABAAB, AAABAAABABBA, AAABAABABAAB, AAABABAABAAB, ABBABAAABAAB, ABABBBABBBABA.
$a \mapsto aaaab$ $b \mapsto abbab$	ABAABBBAAB, AAABBABABB, AAABBABBAB, AABAABBABB, AABABABBBA, AABABBABBA, AABABBBAAB, AABABBBABA, AABBAABBBA, AABBABABBA, AABBABBAAB, AABBABBABA, AABBBAABBA, ABAABBABBA, AABBABABBBA, AABABBBABBBA,
$a \mapsto abb$ $b \mapsto aaab$	AAAA, AAABAABBB, AAABBBABB, AABBABBBA, AABBBABBA, AAABBAAABB,AABABAAABB, ABBBAABBBA, AAABAABBAB, AAABAABAABB, AAABBAABAAB, AABAABAABBA, AABAABBAAAB, AABABABAAAB, AAABBAAABAB, AABAAABABAB, AABAAABBAAB, AAABAABAAABAB,
$a \mapsto aaab$ $b \mapsto bbba$	AAABABBBAA, AAABBAABBB, AAABBABBAA, ABABAAABBB, ABABBBAABBA, AABABBAAABA, AABBABAAABA,
$a \mapsto abaa$ $b \mapsto babb$	AABBABBABBA, AABABBABBBA, AABBBABBBABA, ABABBABBABBA, ABABBABBBABA, ABABBBABABBA, ABBABABBABBA,
$a \mapsto aaaba$ $b \mapsto babbb$	ABAABBBAAA, AABABBBAAA,
$a \mapsto aababbaaaba$ $b \mapsto babbbaababb$	AABAAABAAABAB, ABBBABBBABBBA, AAABAAABAAABAAA,

Over the binary alphabet

morphisms	avoided patterns
$a \mapsto aabaa$ $b \mapsto bbabb$	AABBBAAAB, ABAAABBBA, AAABABABBB, AAABABBABB,AAABABBBAB, AABBBABAAB,AABBBABABA, ABAABABBBA,ABAABBBABA, ABABAABBBA, ABBBABAAAB, AABAABBBAB,AABBBAABAB, AABBBAABAAB, AAABABBAAAB, AABBBABBBAA, ABABABBBABA, ABABBABBABA, AAABAAABBAB, AAABBABAAAB, AAABAABAABAB,AAABABAAABAB, AABAAABABAAB, AAABAAABABBA, AAABAABABAAB, AAABABAABAAB, ABBABAAABAAB, ABABBBABBBABA.
$a \mapsto aaaab$ $b \mapsto abbab$	ABAABBBAAB, AAABBABABB, AAABBABBAB, AABAABBABB, AABABABBBA, AABABBABBA, AABABBBAAB, AABABBBABA, AABBAABBBA, AABBABABBA, AABBABBAAB, AABBABBABA, AABBBAABBA, ABAABBABBA, AABBABABBBA, AABABBBABBBA,
$a \mapsto abb$ $b \mapsto aaab$	AAAA, AAABAABBB, AAABBBABB, AABBABBBA, AABBBABBA, AAABBAAABB,AABABAAABB, ABBBAABBBA, AAABAABBAB, AAABAABAABB, AAABBAABAAB, AABAABAABBA, AABAABBAAAB, AABABABAAAB, AAABBAAABAB, AABAAABABAB, AABAAABBAAB, AAABAABAAABAB,
$a \mapsto aaab$ $b \mapsto bbba$	AAABABBBAA, AAABBAABBB, AAABBABBAA, ABABAAABBB, ABABBBAABBA, AABABBAAABA, AABBABAAABA,
$a \mapsto abaa$ $b \mapsto babb$	AABBABBABBA, AABABBABBBA, AABBBABBBABA, ABABBABBABBA, ABABBABBBABA, ABABBBABABBA, ABBABABBABBA,
$a \mapsto aaaba$ $b \mapsto babbb$	ABAABBBAAA, AABABBBAAA,
$a \mapsto aababbaaaba$ $b \mapsto babbbaababb$	AABAAABAAABAB, ABBBABBBABBBA, AAABAAABAAABAAA,

Lemma

Any binary pattern of length at least 15 contains a pattern from this list.

Over the binary alphabet

morphisms	avoided patterns
$a \mapsto aabaa$ $b \mapsto bbabb$	AABBBAAAB, ABAAABBBA, AAABABABBB, AAABABBABB,AAABABBBAB, AABBBABAAB,AABBBABABA, ABAABABBBA,ABAABBBABA, ABABAABBBA, ABBBABAAAB, AABAABBBAB,AABBBAABAB, AABBBAABAAB, AAABABBAAAB, AABBBABBBAA, ABABABBBABA, ABABBABBABA, AAABAAABBAB, AAABBABAAAB, AAABAABAABAB,AAABABAAABAB, AABAAABABAAB, AAABAAABABBA, AAABAABABAAB, AAABABAABAAB, ABBABAAABAAB, ABABBBABBBABA.
$a \mapsto aaaab$ $b \mapsto abbab$	ABAABBBAAB, AAABBABABB, AAABBABBAB, AABAABBABB, AABABABBBA, AABABBABBA, AABABBBAAB, AABABBBABA, AABBAABBBA, AABBABABBA, AABBABBAAB, AABBABBABA, AABBBAABBA, ABAABBABBA, AABBABABBBA, AABABBBABBBA,
$a \mapsto abb$ $b \mapsto aaab$	AAAA, AAABAABBB, AAABBBABB, AABBABBBA, AABBBABBA, AAABBAAABB,AABABAAABB, ABBBAABBBA, AAABAABBAB, AAABAABAABB, AAABBAABAAB, AABAABAABBA, AABAABBAAAB, AABABABAAAB, AAABBAAABAB, AABAAABABAB, AABAAABBAAB, AAABAABAAABAB,
$a \mapsto aaab$ $b \mapsto bbba$	AAABABBBAA, AAABBAABBB, AAABBABBAA, ABABAAABBB, ABABBBAABBA, AABABBAAABA, AABBABAAABA,
$a \mapsto abaa$ $b \mapsto babb$	AABBABBABBA, AABABBABBBA, AABBBABBBABA, ABABBABBABBA, ABABBABBBABA, ABABBBABABBA, ABBABABBABBA,
$a \mapsto aaaba$ $b \mapsto babbb$	ABAABBBAAA, AABABBBAAA,
$a \mapsto aababbaaaba$ $b \mapsto babbbaababb$	AABAAABAAABAB, ABBBABBBABBBA, AAABAAABAAABAAA,

Lemma

Any binary pattern of length at least 15 contains a pattern from this list.

$\Longrightarrow$ Any binary pattern of length at least 15 is abelian-avoidable over the binary alphabet. $\square$

Open questions

- Can we improve the result with H-D0Ls?
- How to find good morphisms?
- How to show “efficiently” that a pattern is not abelian-avoidable?

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