A language-theoretic approach to elementary number theory

José Manuel Rodríguez Caballero

Laboratoire de combinatoire et d'informatique mathématique (LaCIM)
Université du Québec à Montréal (UQÀM)

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Jacobi θ-function

Let ν be a complex variable on the plane. Let τ be a complex variable on the upper half plane. Let $\theta(\nu)$ be an arbitrary[1] entire function satisfying

$$\theta(\nu + 1) = -\theta(\nu) \text{ and } \theta(\nu + \tau) = -q^{-1} z^{-2} \theta(\nu),$$

provided that $\theta(\nu)$ does not vanish identically, where $z := e^{i\pi\nu}$ and $q := e^{i\pi\tau}$.

1. This function is defined up to a scalar.

Jordan's meromorphic function

The meromorphic function

$$J(\nu) := \frac{\theta'(0)}{2\pi i\theta(\nu)}$$

is well-defined.

Camille Jordan proved the following identity

$$J(\nu) = \frac{1}{z - z^{-1}} + \sum_{m \geq 1} \sum_{n \geq 1} q^{2mn} \left(z^{2m-n} - z^{-2m+n} \right).$$

Jordan's identity

We will use the notation $T_\ell(z)$ for the coefficient of q^ℓ in the expansion of $J(\nu)$. For each even integer $\ell \geq 2$,

$$T_\ell(z) = \sum_{\substack{(m,n) \in (\mathbb{Z}_{\geq 1})^2 \\ 2mn = \ell}} \left(z^{2m-n} - z^{-2m+n} \right).$$

The coefficients of $T_{2n}(z)$, for $n \geq 1$, belong to the set $\{+1, 0, -1\}$.

Welcome to XXI century

In 2011, Tamás Hausel, Emmanuel Letellier and Fernando Rodriguez-Villegas showed a link between the Laurent polynomials[2] $T_\ell(z)$ and the Hodge structure of the Hilbert scheme of n points on $\mathbb{C}^\times \times \mathbb{C}^\times$.

In 2015, Christian Kassel and Christophe Reutenauer showed a link among the Laurent polynomials $T_\ell(z)$, the ideals of the algebra $\mathbb{F}_q[x, y, x^{-1}, y^{-1}]$ and the subgroups of $\mathbb{Z} \times \mathbb{Z}$.

2. The original version was rather different, but equivalent to this one.

The coefficients of $T_{2n}(z)$

Let $n \geq 1$ be an integer. Denote D_n the set of divisors of n. Define $2D_n := \{2d : \quad d \in D_n\}$. Let $\tau_1 < \tau_2 < ... < \tau_k$ be the elements of the symmetric difference $D_n \triangle 2D_n$ written in increasing order. The coefficients of $T_{2n}(z)$ are essentially codified by the word

$$\delta(n) = t_1\, t_2 \ldots t_k,$$

where

$$t_i := \begin{cases} (, & \text{if } \tau_i \in D_n \setminus (2D_n)\,, \\ ), & \text{if } \tau_i \in (2D_n) \setminus D_n. \end{cases}$$

Dyck paths

For any $n \geq 1$, the word $\delta(n)$ is a sequence of balanced parentheses, i.e. the word $\delta(n)$ determines a Dyck path.

Definition

Let Σ be a finite alphabet. Given a set $S \subseteq \mathbb{Z}_{\geq 1}$, we say that S is *rational* (*context-free*) with respect to a function $f : \mathbb{Z}_{\geq 1} \longrightarrow \Sigma^*$, if

$$S = f^{-1}(L)$$

for some rational (context-free) language $L \subseteq \Sigma^*$.

Rational sets with respect to δ

Theorem (J. M. R. C., 2017)

The following sets are rational with respect to δ,

- (i) *the empty set of integers,*
- (ii) *all the integers,*
- (iii) *powers of 2,*
- (iv) *semi-perimeters[a] of Pythagorean triangles.*

a. The semi-perimeter is a half of the perimeter.

Blocks

The *number of blocks* of an integer $n \geq 1$ is defined as the number of connected components of the topological space

$$\bigcup_{d|n} [d, 2d].$$

Context-free sets with respect to δ

Theorem (J. M. R. C., 2017)

The following sets are context-free with respect to δ,

(i) integer having exactly k blocks, for any fixed $k \geq 1$,

(ii) numbers n satisfying $F(n) \geq h$, for any fixed integer $h \geq 1$, where $F(n)$ is the Erdös-Nicolas function[a], i.e.

$$F(n) := \max_{t>0} \# \{d|n : \quad t < d \leq 2t\}.$$

[a]. Paul Erdös, Jean-Louis Nicolas. Méthodes probabilistes et combinatoires en théorie des nombres. Bull. SC. Math **2** (1976) : 301–320.

Definition of $\widehat{\delta}$

For all $n \geq 1$,

$$\widehat{\delta}(n) := \psi\,\delta\,(n)\,,$$

where $\psi : \{((,)), (),)(\}^* \longrightarrow \{A, B, C, D\}^*$ satisfies, for all $w \in \{((,)), (),)(\}^*$,

$$\begin{aligned} \psi\varepsilon &:= \varepsilon, \\ \psi(w) &:= A\psi w, \\ \psi)w(&:= B\psi w, \\ \psi(w(&:= C\psi w, \\ \psi)w) &:= D\psi w. \end{aligned}$$

Hirschhorn function

We define the *Hirschhorn function*[3], $\mathrm{H} : \mathbb{Z}_{\geq 1} \times \{0, 1\} \longrightarrow \mathbb{Z}_{\geq 0}$ by means of the expression

$$\mathrm{H}(n, b) := \#\left\{(a, k) \in \Pi_n : \quad k \equiv b \pmod 2\right\},$$

where Π_n is the set of pairs $(a, k) \in \left(\mathbb{Z}_{\geq 1}\right)^2$ such that

$$n = a + (a + 1) + (a + 2) + ... + (a + k - 1).$$

3. M. D. Hirschhorn and P. M. Hirschhorn. "Partitions into consecutive parts." (2009).

Rational sets with respect to $\hat{\delta}$

Theorem (J. M. R. C., 2017)

Given $k \in \mathbb{Z}_{\geq 0}$ and $b \in \{0, 1\}$, the set of integers $n \geq 1$ satisfying $\mathrm{H}(n, b) \geq k$ is rational with respect to $\hat{\delta}$.

Context-free sets with respect to $\hat{\delta}$

Given $n \geq 1$ and $d|n$, we say that d is a middle divisor of n if

$$\sqrt{\frac{n}{2}} < d \leq \sqrt{2n}.$$

Theorem (J. M. R. C., 2017)

For each integer $k \geq 1$, the set of numbers n having exactly k middle divisors is context-free with respect to $\hat{\delta}$.

We says that n is *odd-trapezoidal* if $\mathrm{H}(n, 0) = 0$.

Theorem (J. M. R. C., 2017)

For all integers $n \geq 2$, if n is not a power of 2 and n is odd-trapezoidal, then $2n$ is the perimeter of a Pythagorean triangle.

Theorem (J. M. R. C., 2017)

For all integers $n \geq 1$, if $2n$ is the perimeter of a Pythagorean triangle, then n has at least two different prime divisors.

We say that n is 2-*densely divisible*[4] if n has only one block.

Theorem (J. M. R. C., 2017)

For all integers $n \geq 2$, if n is 2-densely divisible and $2n$ is not the perimeter of a Pythagorean triangle then n is a power of 2.

4. Densely divisible numbers were introduced by the international team *polymath8*, led by Terence Tao, in order to improve Zhang's bound for the gap between consecutive primes.

The following result is Theorem 3 in Hartmut F. W. Höft, *On the Symmetric Spectrum of Odd Divisors of a Number*, preprint on-line available at `https://oeis.org/A241561/a241561.pdf`

Theorem (Höft, 2015)

For all $n \geq 1$, there exists at least a middle divisor of n if and only if the number of blocks of n is odd.

An open question

A formal language is *decidable* if there exists a total Turing machine[5] that, when given a finite sequence of symbols as input, accepts it if it belongs to the language and rejects it otherwise.

Is the language $\delta\left(\mathbb{Z}_{\geq 1}\right)$ decidable ?

5. A total Turing machine is a Turing machine that halts for every given input.

Theoretical extensions

Recently I found some connections between the coefficients of $T_\ell(-z)$ and the lengths of the hypotenuses of primitive Pythagorean triangles[6].

It seem that there is an arithmetical theory for the words encoding the coefficients of $T_\ell(-z)$.

6. Pythagorean triangles for which the length of sides are relative primes.

Coq-AI project (Science Fiction ?)

(i) To formalize this theory as a library in the proof assistant Coq.

(ii) To combine Coq with artificial intelligence in order to obtain "interesting" theorems [7] in an autonomous way, e.g., using genetic algorithms or neuronal networks.

(iii) To check whether or not the theorems obtained by the computer are essentially the same as the theorems showed in this talk.

(iv) Next time my computer will give my talk.

7. We need to find a criterion for a theorem to be interesting. Some notion of **complexity** should be involved.

Main References

- Tamás Hausel, Emmanuel Letellier and Fernando Rodriguez-Villegas. *Arithmetic harmonic analysis on character and quiver varieties*, Duke Mathematical Journal **160**.2 (2011) : 323-400.
- Tamás Hausel, Emmanuel Letellier and Fernando Rodriguez-Villegas. *Arithmetic harmonic analysis on character and quiver varieties II*, Advances in Mathematics **234** (2013) : 85-128.
- Christian Kassel and Christophe Reutenauer, *The Fourier expansion of* $\eta(z)\,\eta(2z)\,\eta(3z)\,/\,\eta(6z)$, Archiv der Mathematik **108**.5 (2017) : 453-463.
- Christian Kassel and Christophe Reutenauer, *Counting the ideals of given codimension of the algebra of Laurent polynomials in two variables*, Michigan J Maths (to appear).
- Christian Kassel and Christophe Reutenauer, *Complete determination of the zeta function of the Hilbert scheme of n points on a two-dimensional torus*, Ramanujan Journal (to appear).

My publications related to this subject

- José Manuel Rodríguez Caballero "Symmetric Dyck Paths and Hooley's Δ-Function." **International Conference on Combinatorics on Words. Springer**, Cham, 2017.
- José Manuel Rodríguez Caballero. "Divisors on Overlapped Intervals and Multiplicative Functions." **Journal of Integer Sequences** 20.2 (2017) : 3.
- José Manuel Rodríguez Caballero. "On Kassel–Reutenauer q-analog of the sum of divisors and the ring $\mathbb{F}_3[X]/X^2\mathbb{F}_3[X]$." **Finite Fields and Their Applications. Elsevier** 51 (2018) : 183-190.

Thank you ! !