Games and multidimensional shape-symmetric morphisms

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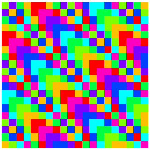
Workshop on Words and Complexity
Lyon, 19th February 2018

Université de Liège



Why this talk?

- I started working with Eric Duchêne more than 10 years ago!
- I recently gave a course at CIRM
  - A video is available `http://library.cirm-math.fr/`
  - A chapter is on its way...
- Nice applications of combinatorics on words
- young researchers attending this workshop

- **MR3621222** Reviewed Duchêne, Eric; Parreau, Aline; Rigo, Michel Deciding game invariance. *Inform. and Comput.* 253 (2017), part 1, 127–142. 91A46 (03B25 68Q45) Get it@ULiège
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- **MR3544849** Reviewed Cassaigne, Julien; Duchêne, Eric; Rigo, Michel Nonhomogeneous Beatty sequences leading to invariant games. *SIAM J. Discrete Math.* 30 (2016), no. 3, 1798–1829. 91A05 (11B83 11P81 68R15 91A46) Get it@ULiège
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- **MR2676861** Reviewed Duchêne, Eric; Rigo, Michel Invariant games. *Theoret. Comput. Sci.* 411 (2010), no. 34-36, 3169–3180. (Reviewer: Paweł Prałat) 91A46 Get it@ULiège
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- **MR2600974** Reviewed Duchêne, Eric; Fraenkel, Aviezri S.; Nowakowski, Richard J.; Rigo, Michel Extensions and restrictions of Wythoff's game preserving its $\mathscr{P}$ positions. *J. Combin. Theory Ser. A* 117 (2010), no. 5, 545–567. (Reviewer: Thane Earl Plambeck) 91A46 (91A43) Get it@ULiège
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- **MR2461578** Reviewed Duchêne, Eric; Rigo, Michel Cubic Pisot unit combinatorial games. *Monatsh. Math.* 155 (2008), no. 3-4, 217–249. (Reviewer: Petr Ambrož) 68R15 (11A67 91A05 91A46) Get it@ULiège
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- **MR2401268** Reviewed Duchêne, Eric; Rigo, Michel A morphic approach to combinatorial games: the Tribonacci case. *Theor. Inform. Appl.* 42 (2008), no. 2, 375–393. (Reviewer: Narad Rampersad) 91A46 (68Q45 68R15) Get it@ULiège
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Wythoff's game or, *the Queen* ♕ *goes to* $(0,0)$

- two players playing alternatively;
- the player *unable to move* loses the game (Normal play);
- two piles of token;
- Nim rule : remove a positive number of token from one pile ♖

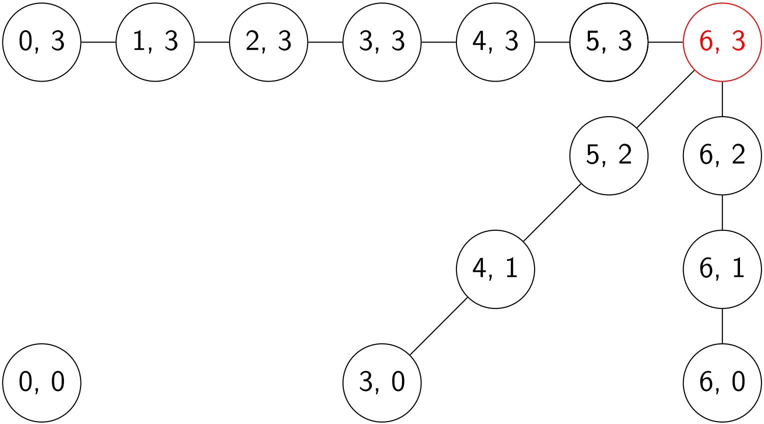
$$\text{Moves} = \{(i,0),(0,i) \mid i \geq 1\}.$$

- Wythoff's rule: remove *simultaneously* the same number of token from both piles

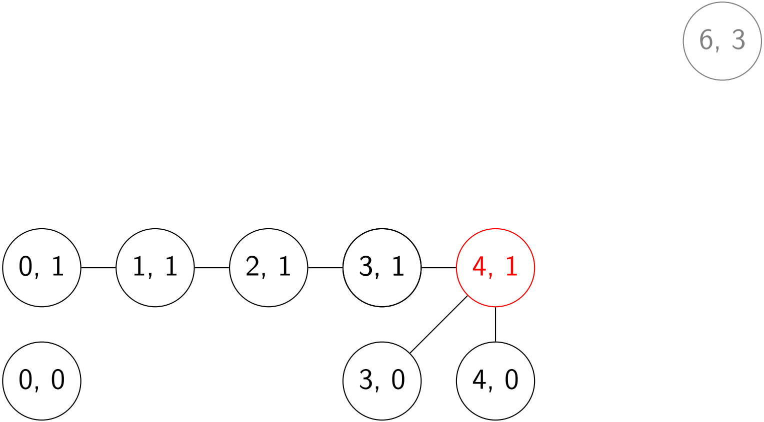
$$\text{Moves} = \{(i,0),(0,i),(i,i) \mid i \geq 1\}.$$

♕ $(6, 3)$

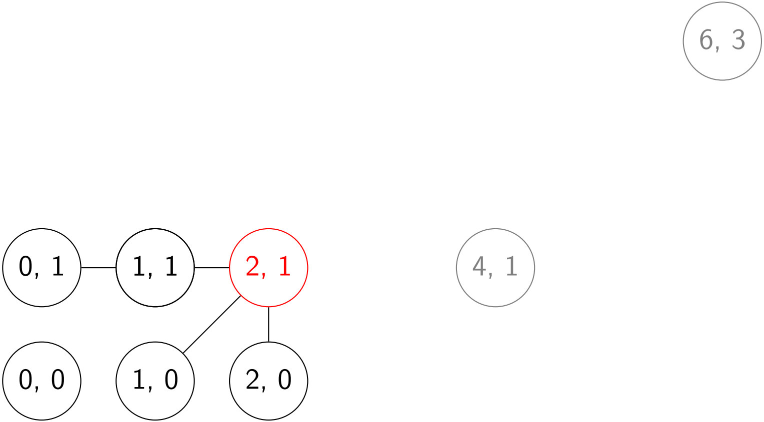
♕ $(6, 3)$



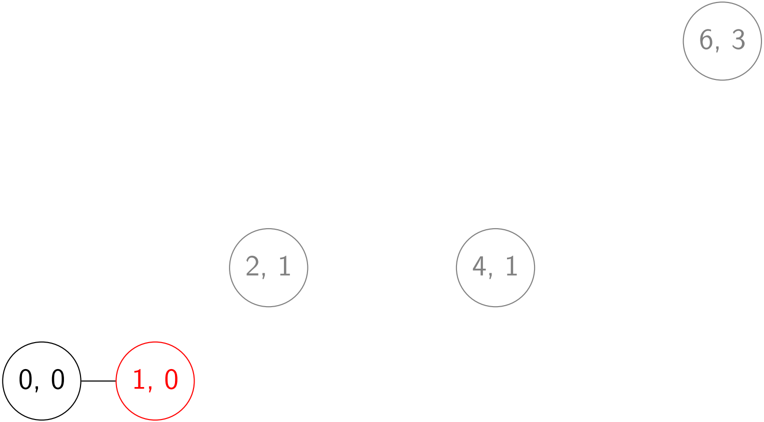
♕ $(6, 3) \xrightarrow{A} (4, 1)$



♛ $(6,3) \xrightarrow{A} (4,1) \xrightarrow{B} (2,1)$



$$♕\ (6,3) \xrightarrow{A} (4,1) \xrightarrow{B} (2,1) \xrightarrow{A} (1,0)$$



♕ $$(6,3) \xrightarrow{A} (4,1) \xrightarrow{B} (2,1) \xrightarrow{A} (1,0) \xrightarrow{B} (0,0)$$

Winning and losing positions:

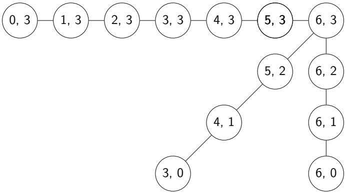
STATUS $\mathcal{N}$ (NEXT MOVE) OR $\mathcal{P}$ (PREVIOUS PLAYER)

A position is $\mathcal{P}$, if all its options are $\mathcal{N}$;
A position is $\mathcal{N}$, if there exists an option in $\mathcal{P}$.

If the *game-graph* is acyclic

- vertices = positions
- edges = available options,

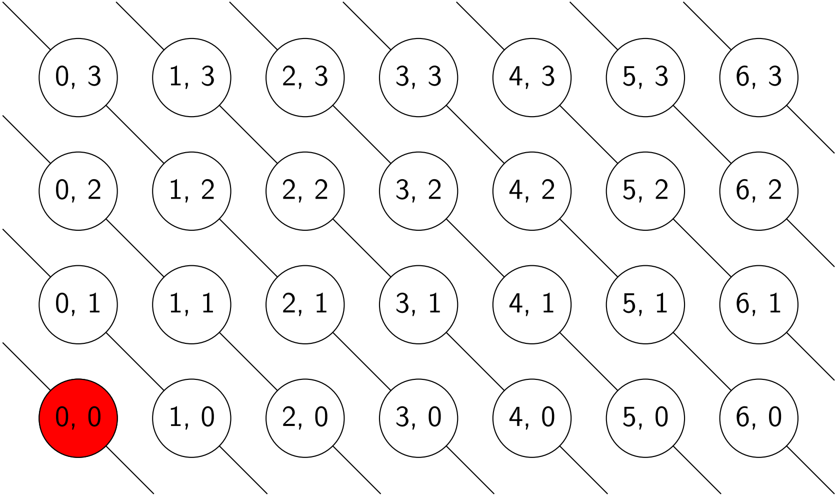
every position is either $\mathcal{N}$, or $\mathcal{P}$.



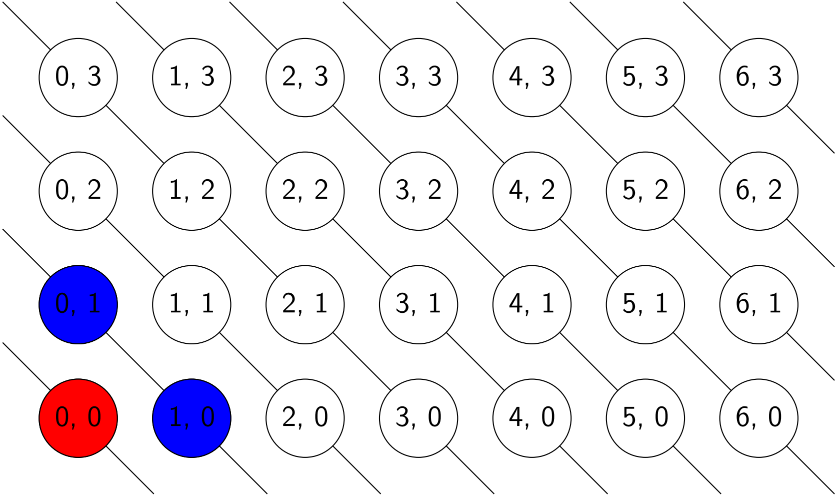
Remark (graph-theoretic notion)

- The set of $\mathcal{P}$-positions is the *kernel* of the game-graph:
  - stable set: $k \not\longrightarrow k'$;
  - absorbing set: $\ell \longrightarrow k$;
  - always exists for acyclic graphs.
- The game-graph grows exponentially.

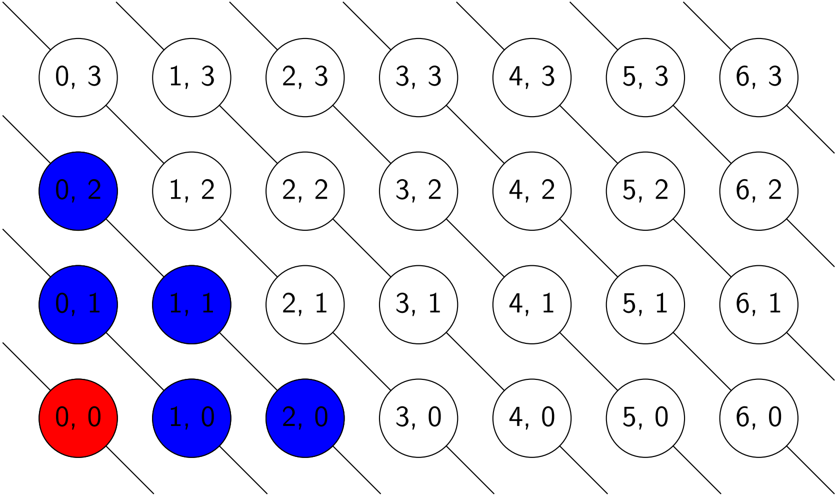
A *winning strategy* is a map from $\mathcal{N}$ to $\mathcal{P}$ assigning to every winning position in $\mathcal{N}$ an available option in $\mathcal{P}$.



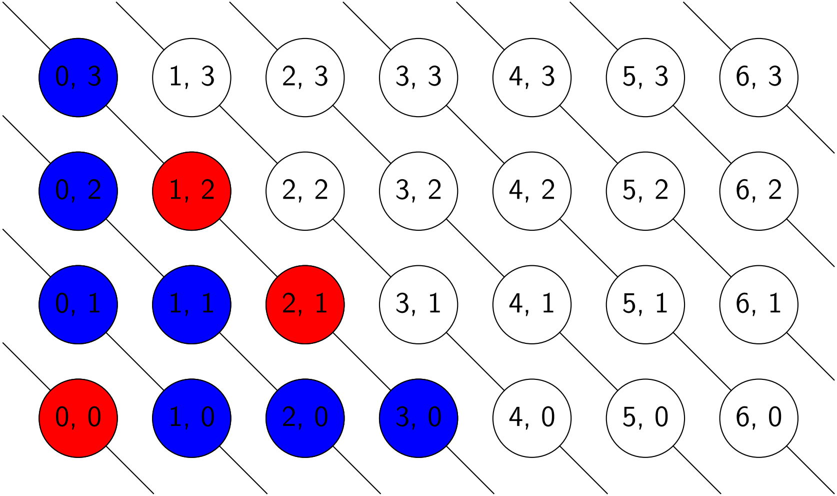
$\mathcal{P}$-positions and $\mathcal{N}$-positions for Wythoff's game.



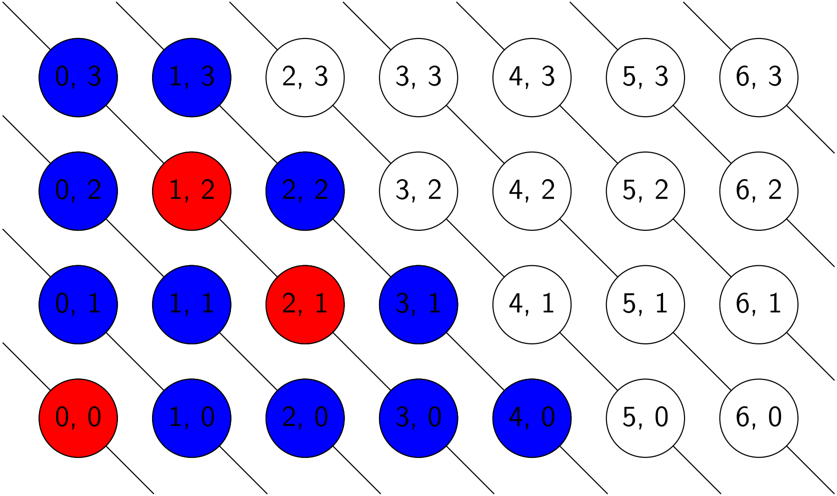
$\mathcal{P}$-positions and $\mathcal{N}$-positions for Wythoff's game.



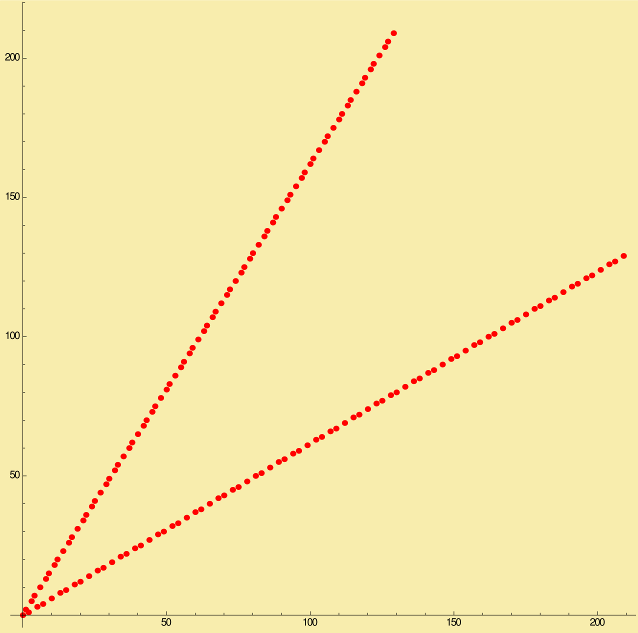
$\mathcal{P}$-positions and $\mathcal{N}$-positions for Wythoff's game.



$\mathcal{P}$-positions and $\mathcal{N}$-positions for Wythoff's game.



$\mathcal{P}$-positions and $\mathcal{N}$-positions for Wythoff's game.



$\mathcal{P}$-positions and $\mathcal{N}$-positions for Wythoff's game.

Definition

Let $S \subset \mathbb{N}$. MeX (minimum excluded value) of $S = \min \mathbb{N} \setminus S$.

Let G be a combinatorial game and x be a position.
The *Grundy function* is given by

$$\mathcal{G}(x) = \mathrm{MeX}(\mathcal{G}(\mathrm{Opt}(x))).$$

$$\mathrm{MeX}\{0,1,3,5\} = 2, \quad \mathrm{MeX}\{2,3,6\} = 0, \quad \mathrm{MeX}\emptyset = 0.$$

Characterization of the $\mathcal{P}$-positions

Let x be a position. We have $\mathcal{G}(x) = 0$ iff x is in $\mathcal{P}$.

Nim on one pile

$\mathcal{G}(p) = p$ where p is the number of token left.

Theorem (Sprague–Grundy)

Let G_i be combinatorial games with $\mathcal{G}_i$ as Grundy function, $i = 1, \ldots, n$. Then the disjunctive sum of games $G_1 + \cdots + G_n$ has Grundy function

$$\mathcal{G}(x_1, \ldots, x_n) = \mathcal{G}_1(x_1) \oplus \cdots \oplus \mathcal{G}_n(x_n)$$

where $\oplus$ is the Nim-sum.

Nim on n piles is the sum of n games of Nim on one pile.

Application

Let's play on four boards simultaneously:

- G_1 Nim $\mathcal{G}_1(2, 5) = 7$
- G_2 Wythoff $\mathcal{G}_2(3, 4) = 2$
- G_3 Nim on three piles $\mathcal{G}_3(8, 7, 6) = 9$
- G_4 Wythoff $\mathcal{G}_4(3, 9) = 12$

Should you start? Just compute whether $7 \oplus 2 \oplus 9 \oplus 12$ is 0 or not?

General questions

- Characterize the set of $\mathcal{P}$-positions?
- Is it computationally hard to determine these positions?
- Compute a winning strategy.

Thanks to Sprague–Grundy theorem, we have an extra motivation:

- *Compute the Grundy function of all positions.*

For the game of Nim, first few values of $(x, y) \mapsto \mathcal{G}_N(x, y) = x \oplus y$

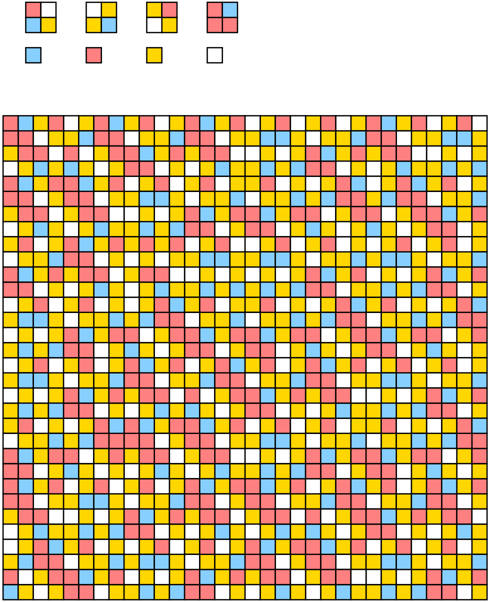
$\vdots$	$\vdots$										$\iddots$
9	9	8	11	10	13	12	15	14	1	0	
8	8	9	10	11	12	13	14	15	0	1	
7	7	6	5	4	3	2	1	0	15	14	
6	6	7	4	5	2	3	0	1	14	15	
5	5	4	7	6	1	0	3	2	13	12	
4	4	5	6	7	0	1	2	3	12	13	
3	3	2	1	0	7	6	5	4	11	10	
2	2	3	0	1	6	7	4	5	10	11	
1	1	0	3	2	5	4	7	6	9	8	
0	0	1	2	3	4	5	6	7	8	9	$\cdots$
	0	1	2	3	4	5	6	7	8	9	$\cdots$

$\rightsquigarrow$ Exercises 21 and 22 in Section 16.6, p.451, Allouche–Shallit'03.

Regular sequences

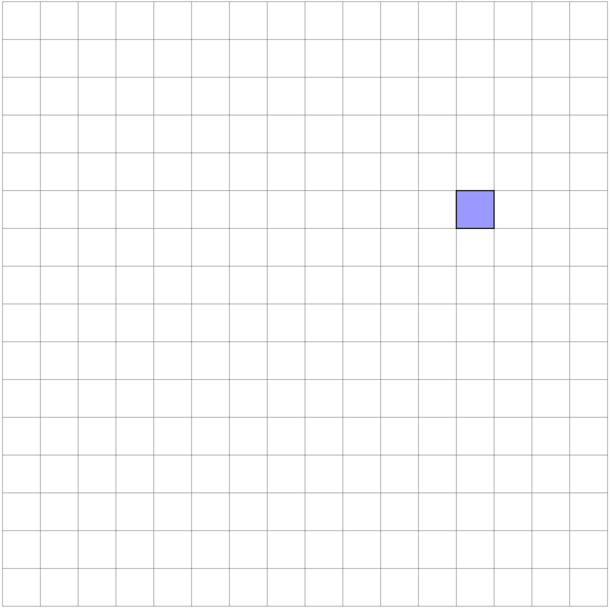
What can be said about the structure of this table?

- Let us start with multidimensional k-automatic sequences;
- then move to k-regular sequences.



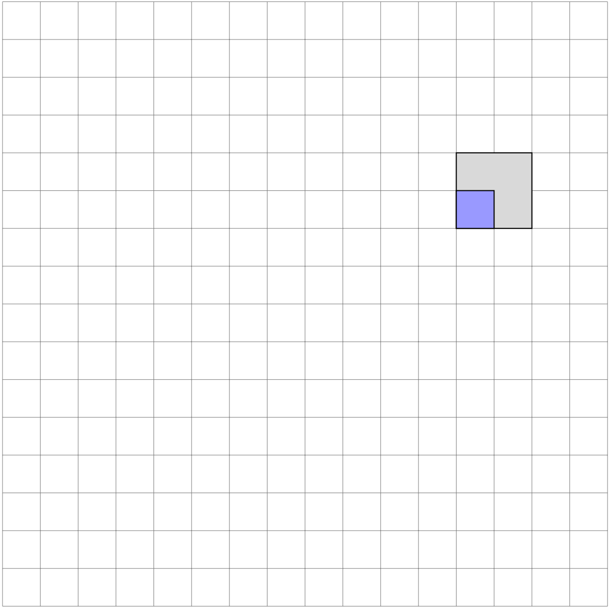
O. Salon, Suites automatiques à multi-indices, *Séminaire de théorie des nombres*, Bordeaux, 1986–1987, exposé 4.

Tracking the past



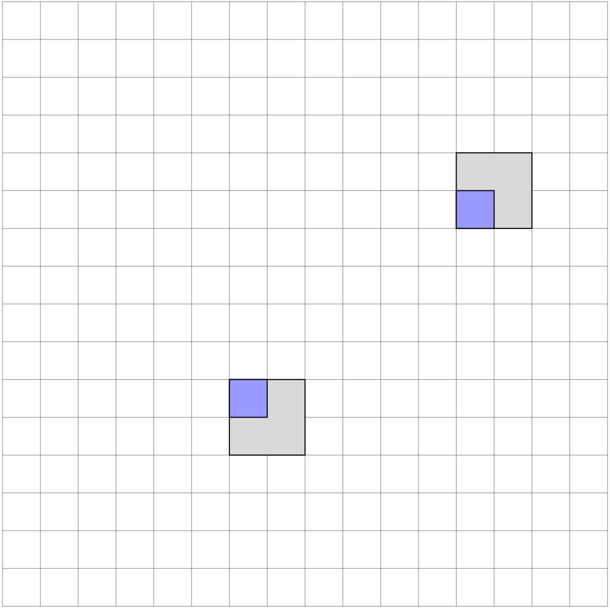
$x(12, 10) \quad \mathrm{rep}_2(12) = 1100, \quad \mathrm{rep}_2(10) = 1010$

Tracking the past



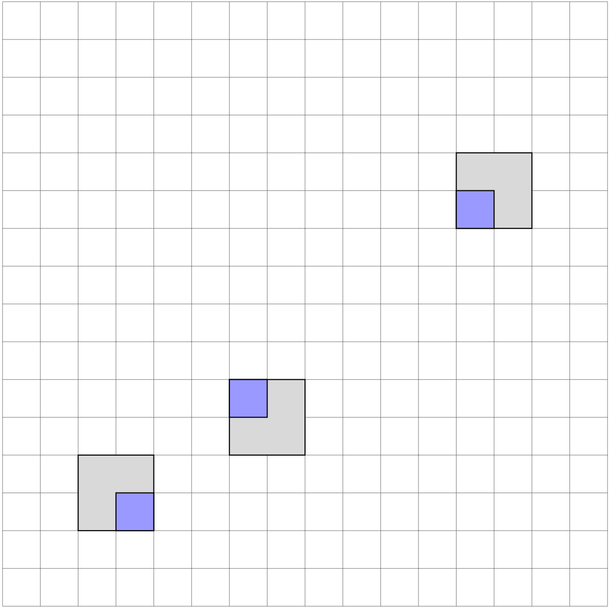
$$x(12, 10) \quad \mathrm{rep}_2(12) = 1100, \quad \mathrm{rep}_2(10) = 1010$$

Tracking the past



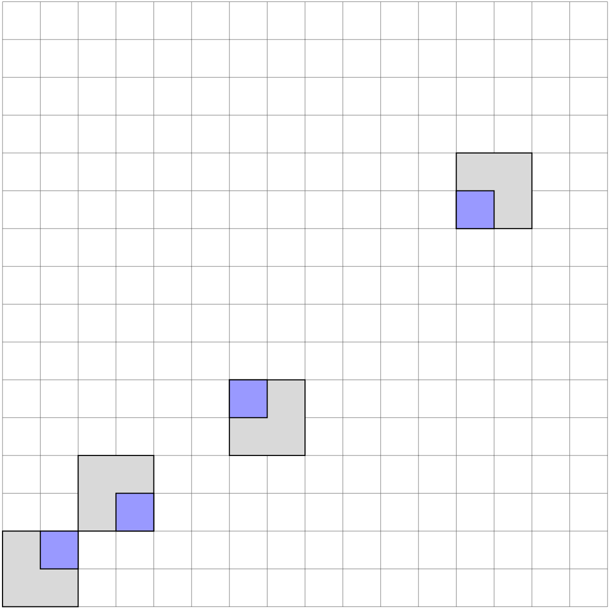
$x(6,5) \to x(12,10) \quad \text{rep}_2(6) = 110, \quad \text{rep}_2(5) = 101$

Tracking the past



$$x(3,2) \to x(6,5) \to x(12,10) \quad \mathrm{rep}_2(3) = 11, \quad \mathrm{rep}_2(2) = 10$$

Tracking the past



$x(1,1) \to x(3,2) \to x(6,5) \to x(12,10) \quad \mathrm{rep}_2(3) = 1, \quad \mathrm{rep}_2(2) = 1$

Definition of the k-kernel in a multidimensional setting

Definition

Consider a bi-dimensional sequence $\mathbf{x} = (x(m,n))_{m,n\geq 0}$.
It is a set of bi-dimensional subsequences:

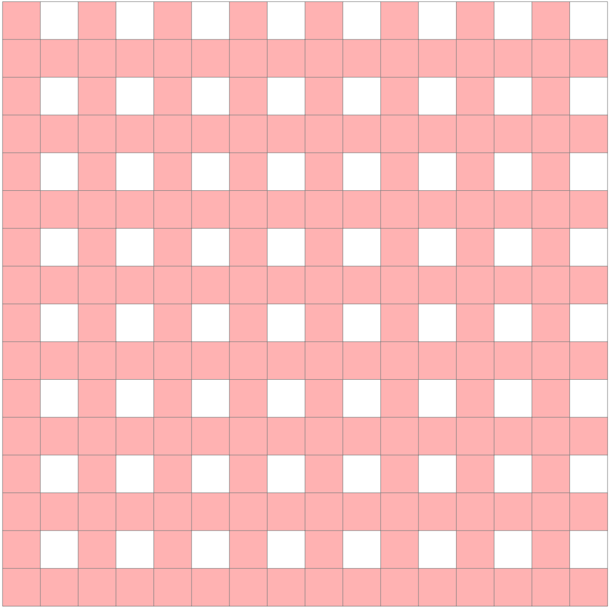
$$\mathrm{Ker}_k(\mathbf{x}) = \left\{(x(k^i m + r, k^i n + s))_{m,n\geq 0} \mid i \geq 0, 0 \leq r, s < k^i\right\}.$$

This corresponds to selecting the suffixes

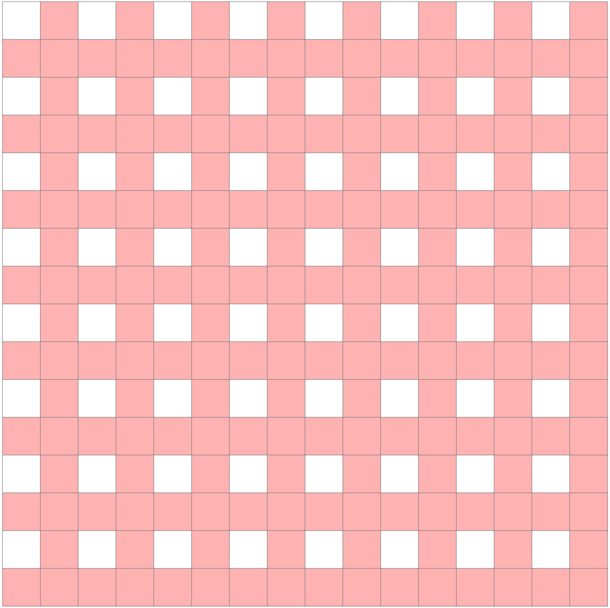
$$(0^{i-p} r_p \cdots r_1, 0^{i-q} s_q \cdots s_1)$$

where $\mathrm{rep}_k(r) = r_p \cdots r_1$ and $\mathrm{rep}_k(s) = s_q \cdots s_1$.

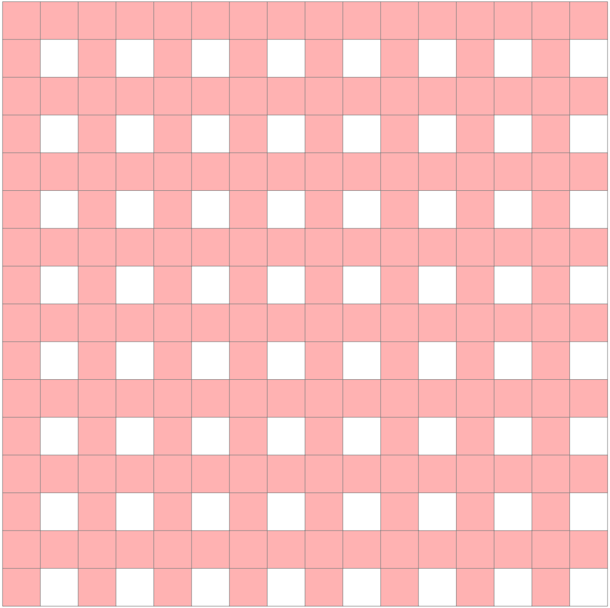
Some of these subsequences $(0,0)$



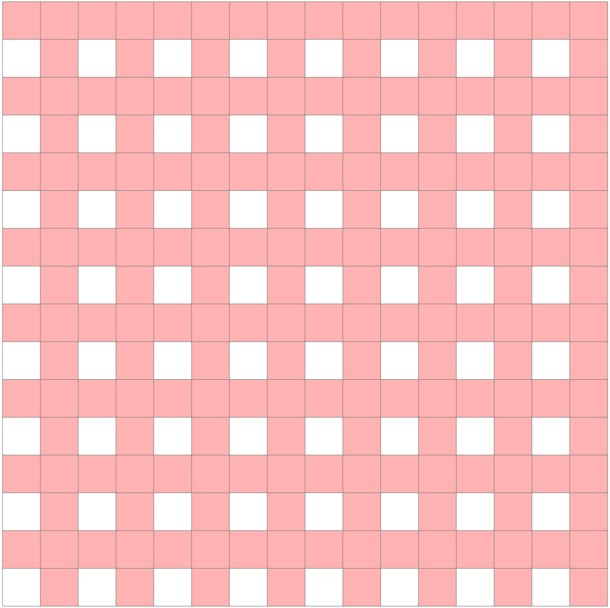
Some of these subsequences $(1, 0)$



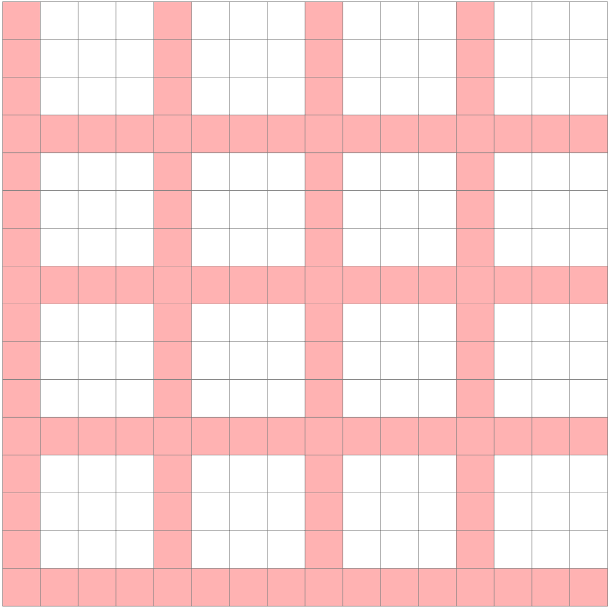
Some of these subsequences $(0, 1)$



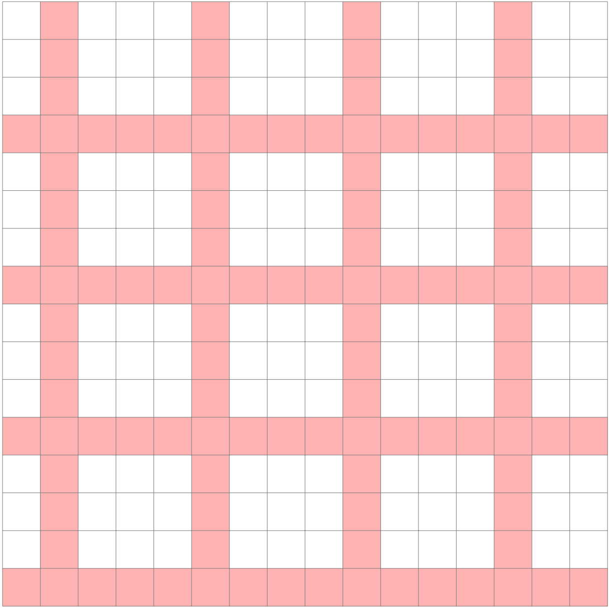
Some of these subsequences $(1,1)$



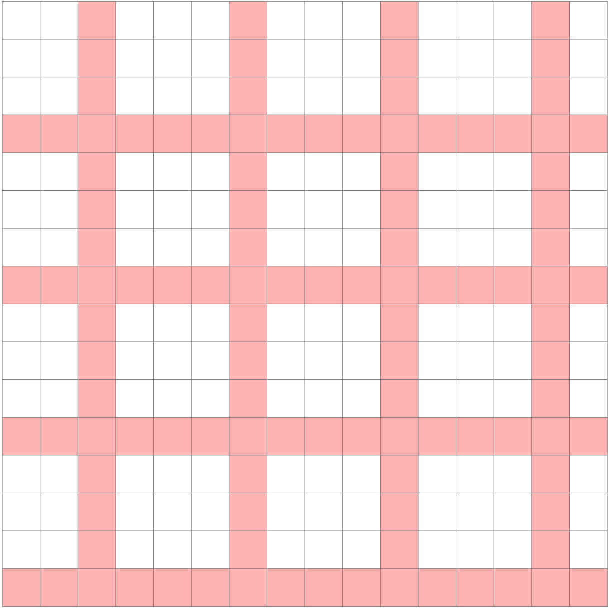
Some of these subsequences $(00, 00)$



Some of these subsequences $(01, 00)$



Some of these subsequences $(10, 00)$



$\rightsquigarrow$ We can define multidimensional k-regular sequences.
The $\mathbb{Z}$-module generated by $\mathrm{Ker}_k(\mathbf{x})$ is finitely generated.

Proposition (Exercise)

For the game of Nim, $(\mathcal{G}_N(m,n))_{m,n\geq 0}$ is 2-regular.

Proof. We have

$$\begin{aligned}
\mathcal{G}_N(2m,2n) &= & 2m \oplus 2n & &= 2\,\mathcal{G}_N(m,n)\\
\mathcal{G}_N(2m+1,2n) &= & (2m+1) \oplus 2n & &= 2\,\mathcal{G}_N(m,n)+1\\
\mathcal{G}_N(2m,2n+1) &= & 2m \oplus (2n+1) & &= 2\,\mathcal{G}_N(m,n)+1\\
\mathcal{G}_N(2m+1,2n+1) &= & (2m+1) \oplus (2n+1) & &= 2\,\mathcal{G}_N(m,n)
\end{aligned}$$

thus the 2-kernel is generated by $(\mathcal{G}_N(m,n))_{m,n\geq 0}$ and the constant sequence (1). $\square$

Is that clear for any element of the 2-kernel?
Can $(\mathcal{G}_N(8m+5, 8n+2))_{m,n\geq 0}$ be expressed as a $\mathbb{Z}$-linear combination of these two sequences?

$$\begin{aligned}
\mathcal{G}_N(8m+5, 8n+2) &= \mathcal{G}_N\left(2(4m+2)+1, 2(4n+1)\right) \\
&= 2\,\mathcal{G}_N(4m+2, 4n+1)+1 \\
&= 2\,\mathcal{G}_N(2(2m+1), 2.2n+1)+1 \\
&= 2\left[2\,\mathcal{G}_N(2m+1, 2n)+1\right]+1 \\
&= 4\,\mathcal{G}_N(2m+1, 2n)+3 \\
&= 4\left[2\,\mathcal{G}_N(m, n)+1\right]+3 \\
&= 8\,\mathcal{G}_N(m, n)+7.
\end{aligned}$$

Meaning of these relations within the table:

9	8	11	10	13	12	15	14	1	0
8	9	10	11	12	13	14	15	0	1
7	6	5	4	3	2	1	0	15	14
6	7	4	5	2	3	0	1	14	15
5	4	7	6	1	0	3	2	13	12
4	5	6	7	0	1	2	3	12	13
3	2	1	0	7	6	5	4	11	10
2	3	0	1	6	7	4	5	10	11
1	0	3	2	5	4	7	6	9	8
0	1	2	3	4	5	6	7	8	9

First few values of $\mathcal{G}_N(m,n)$.

$$\mathcal{G}_N(m,n) \mapsto \begin{array}{|c|c|}\hline 2\,\mathcal{G}_N(m,n)+1 & 2\,\mathcal{G}_N(m,n) \\ \hline 2\,\mathcal{G}_N(m,n) & 2\,\mathcal{G}_N(m,n)+1 \\ \hline \end{array}$$

For the game of Wythoff, first few values of $(x, y) \mapsto \mathcal{G}_W(x, y)$

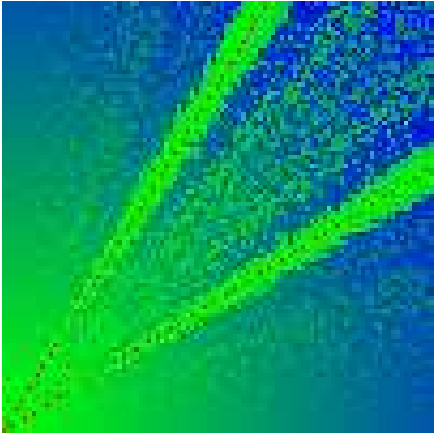
$\vdots$											$\iddots$
9	9	10	11	12	8	7	13	14	15	16	
8	8	6	7	10	1	2	5	3	4	15	
7	7	8	6	9	0	1	4	5	3	14	
6	6	7	8	1	9	10	3	4	5	13	
5	5	3	4	0	6	8	10	1	2	7	
4	4	5	3	2	7	6	9	0	1	8	
3	3	4	5	6	2	0	1	9	10	12	
2	2	0	1	5	3	4	8	6	7	11	
1	1	2	0	4	5	3	7	8	6	10	
0	0	1	2	3	4	5	6	7	8	9	$\cdots$
	0	1	2	3	4	5	6	7	8	9	$\cdots$

For Wythoff's game, not so many results are known

- U. Blass, A.S. Fraenkel, The Sprague-Grundy function for Wythoff's game. Theoret. Comput. Sci. 75 (1990), no. 3, 311–333.
- Y. Jiao, On the Sprague-Grundy values of the $\mathcal{F}$-Wythoff game. Electron. J. Combin. 20 (2013).
- A. Gu, Sprague-Grundy values of the $\mathcal{R}$-Wythoff game. Electron. J. Combin. 22 (2015).
- M. Weinstein, Invariance of the Sprague-Grundy function for variants of Wythoff's game. Integers 16 (2016).

It's challenging, we quote the Siegel's book:

"*No general formula is known for computing arbitrary $\mathcal{G}$-values of WYTHOFF. In general, they appear chaotic, though they exhibit a striking fractal-like pattern ... Despite this apparent chaos, the $\mathcal{G}$-values nonetheless have a high degree of geometric regularity.*"



$\mathcal{G}_W(m,n),\ m,n \leq 100$

Proposition (Allouche–Shallit)

The projection on a finite alphabet of a k-regular sequence is a k-automatic sequence.

Shape-symmetric morphisms

Question: *What can be said about the (morphic) structure of the $\mathcal{P}$-positions of Wythoff's ♕ game?*

$$
(P_{i,j})_{i,j\geq 0} = \left|\begin{array}{cccccccccccc}
\vdots & & & & & & & & & & & \iddots \\
0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\
0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & \\
0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & \\
0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & \\
0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\
0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\
\mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
\hline
\end{array}\right.
$$

Let's try something...

$$\varphi_W : a \mapsto \begin{array}{|c|c|}\hline c & d \\ \hline a & b \\ \hline\end{array} \quad b \mapsto \begin{array}{|c|}\hline e \\ \hline i \\ \hline\end{array} \quad c \mapsto \begin{array}{|c|c|}\hline i & j \\ \hline\end{array} \quad d \mapsto \begin{array}{|c|}\hline i \\ \hline\end{array} \quad e \mapsto \begin{array}{|c|c|}\hline f & b \\ \hline\end{array}$$

$$f \mapsto \begin{array}{|c|c|}\hline h & d \\ \hline g & b \\ \hline\end{array} \quad g \mapsto \begin{array}{|c|c|}\hline h & d \\ \hline f & b \\ \hline\end{array} \quad h \mapsto \begin{array}{|c|c|}\hline i & m \\ \hline\end{array} \quad i \mapsto \begin{array}{|c|c|}\hline h & d \\ \hline i & m \\ \hline\end{array}$$

$$j \mapsto \begin{array}{|c|}\hline c \\ \hline k \\ \hline\end{array} \quad k \mapsto \begin{array}{|c|c|}\hline c & d \\ \hline l & m \\ \hline\end{array} \quad l \mapsto \begin{array}{|c|c|}\hline c & d \\ \hline k & m \\ \hline\end{array} \quad m \mapsto \begin{array}{|c|}\hline h \\ \hline i \\ \hline\end{array}$$

and the coding

$$\mu_W : a, e, g, j, l \mapsto 1, \quad b, c, d, f, h, i, k, m \mapsto 0$$

Let $d \geq 2$

A d-dimensional picture over A is a map

$$x : [\![0, s_1 - 1]\!] \times \cdots \times [\![0, s_d - 1]\!] \to A$$

$(s_1, \ldots, s_d)$ is the shape of x; if $s_i < \infty$, for all i, x is bounded. The set of bounded pictures over A is denoted by $\mathcal{B}_d(A)$.

If for some $i \in [\![1, d]\!]$, $|x|_{\widehat{i}} = |y|_{\widehat{i}} = (s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_d)$, then we define the concatenation of x and y *in the direction* i to be the d-dimensional picture $x \odot^i y$ of shape

$$(s_1, \ldots, s_{i-1}, |x|_i + |y|_i, s_{i+1}, \ldots, s_d).$$

An example

$$x = \begin{array}{|c|c|}\hline a & b \\ \hline c & d \\ \hline\end{array} \quad \text{and} \quad y = \begin{array}{|c|c|c|}\hline a & a & b \\ \hline b & c & d \\ \hline\end{array}$$

of shape respectively $|x| = (2,2)$ and $|y| = (2,3)$.

Since $|x|_{\widehat{2}} = |y|_{\widehat{2}} = 2$, we get

$$x \odot^2 y = \begin{array}{|c|c|c|c|c|}\hline a & b & a & a & b \\ \hline c & d & b & c & d \\ \hline\end{array}\,.$$

However $x \odot^1 y$ is not defined because $2 = |x|_{\widehat{1}} \neq |y|_{\widehat{1}} = 3$.

Remark

A map $\gamma\colon A \to \mathcal{B}_d(A)$ cannot necessarily be extended to a morphism $\gamma\colon \mathcal{B}_d(A) \to \mathcal{B}_d(A)$.

$$\gamma: \ a \mapsto \begin{array}{|c|c|}\hline b & d \\ \hline a & a \\ \hline\end{array}, \quad b \mapsto \begin{array}{|c|}\hline b \\ \hline c \\ \hline\end{array}, \quad c \mapsto \begin{array}{|c|c|}\hline a & a \\ \hline\end{array}, \quad d \mapsto \begin{array}{|c|}\hline d \\ \hline\end{array}.$$

$$\odot^2\colon \ |\gamma(c)|_{\widehat{2}} = |\gamma(d)|_{\widehat{2}} = 1, \quad |\gamma(a)|_{\widehat{2}} = |\gamma(b)|_{\widehat{2}} = 2,$$
$$\odot^1\colon \ |\gamma(a)|_{\widehat{1}} = |\gamma(c)|_{\widehat{1}} = 2, \quad |\gamma(d)|_{\widehat{1}} = |\gamma(b)|_{\widehat{1}} = 1.$$

$$x = \begin{array}{|c|c|}\hline c & d \\ \hline a & b \\ \hline\end{array}, \quad \gamma(x) = \begin{array}{|c|c|c|}\hline a & a & d \\ \hline b & d & b \\ \hline a & a & c \\ \hline\end{array}$$

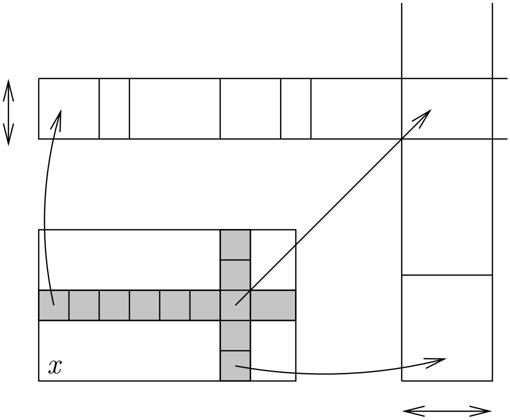
$$\gamma:\ a \mapsto \begin{array}{|c|c|}\hline b & d \\ \hline a & a \\ \hline\end{array}\,, \quad b \mapsto \begin{array}{|c|}\hline b \\ \hline c \\ \hline\end{array}\,, \quad c \mapsto \begin{array}{|c|c|}\hline a & a \\ \hline\end{array}\,, \quad d \mapsto \begin{array}{|c|}\hline d \\ \hline\end{array}\,.$$

$$x = \begin{array}{|c|c|}\hline c & d \\ \hline a & b \\ \hline\end{array}\,, \quad \gamma(x) = \begin{array}{|c|c|c|}\hline a & a & d \\ \hline b & d & b \\ \hline a & a & c \\ \hline\end{array}$$

but $\gamma^2(x)$ is not well-defined:

$$\gamma^2(x) \rightsquigarrow \begin{array}{ccc}
\begin{array}{|c|c|}\hline b & d \\ \hline a & a \\ \hline\end{array} & \begin{array}{|c|c|}\hline b & d \\ \hline a & a \\ \hline\end{array} & \begin{array}{|c|}\hline d \\ \hline\end{array} \\
\begin{array}{|c|}\hline b \\ \hline c \\ \hline\end{array} & \begin{array}{|c|}\hline d \\ \hline\end{array} & \begin{array}{|c|}\hline b \\ \hline c \\ \hline\end{array} \\
\begin{array}{|c|c|}\hline b & d \\ \hline a & a \\ \hline\end{array} & \begin{array}{|c|c|}\hline b & d \\ \hline a & a \\ \hline\end{array} & \begin{array}{|c|c|}\hline a & a \\ \hline\end{array}
\end{array}$$

What do we need for $\gamma(x)$ to be defined?



$\rightsquigarrow$ the images of any two symbols on a row (resp. column) have the same number of rows (resp. columns).

In a formal way ($\star$)

Let $\gamma\colon A \to \mathcal{B}_d(A)$ be a map and x be a bounded d-dimensional picture such that

$$\forall i \in \{1, \ldots, d\}, \forall k < |x|_i, \forall a, b \in \mathsf{Alph}(x_{|i,k}) : |\gamma(a)|_i = |\gamma(b)|_i.$$

$\mathsf{Alph}(x_{|i,k})$ is the set of letters occurring in the section $x_{|i,k}$.

Then the *image* of x by γ is the d-dimensional picture defined as

$$\gamma(x) = \odot^1_{0 \le n_1 < |x|_1} \left(\cdots \left(\odot^d_{0 \le n_d < |x|_d} \gamma(x(n_1, \ldots, n_d)) \right) \cdots \right).$$

Definition

If for all $a \in A$ and all $n \geq 1$, $\gamma^n(a)$ is well-defined from $\gamma^{n-1}(a)$, then γ is said to be a d-dimensional morphism. We can define accordingly a prolongable morphism.

Definition

Let $\gamma \colon \mathcal{B}_d(A) \to \mathcal{B}_d(A)$ be a d-dimensional morphism having the d-dimensional infinite word $\mathbf{x}$ as a fixed point.

This word is shape-symmetric with respect to γ if, for all permutations ν of $[\![1, d]\!]$, we have, for all $n_1, \ldots, n_d \geq 0$,

$$|\gamma(\mathbf{x}(n_1, \ldots, n_d))| = (s_1, \ldots, s_d)$$

$$\Downarrow$$

$$|\gamma(\mathbf{x}(n_{\nu(1)}, \ldots, n_{\nu(d)}))| = (s_{\nu(1)}, \ldots, s_{\nu(d)}).$$

Reconsider our map φ_W (one can indeed prove that it is a d-dimensional morphism having a shape-symmetric fixed point).

$$a \mapsto \begin{array}{|c|c|}\hline c & d \\ \hline \mathbf{a} & b \\ \hline\end{array} \mapsto \begin{array}{|c|c|c|}\hline i & \mathbf{j} & i \\ \hline c & d & \mathbf{e} \\ \hline \mathbf{a} & b & i \\ \hline\end{array} \mapsto \begin{array}{|c|c|c|c|c|}\hline h & d & c & h & d \\ \hline i & m & k & i & m \\ \hline i & \mathbf{j} & i & f & b \\ \hline c & d & \mathbf{e} & h & d \\ \hline \mathbf{a} & b & i & i & m \\ \hline\end{array}$$

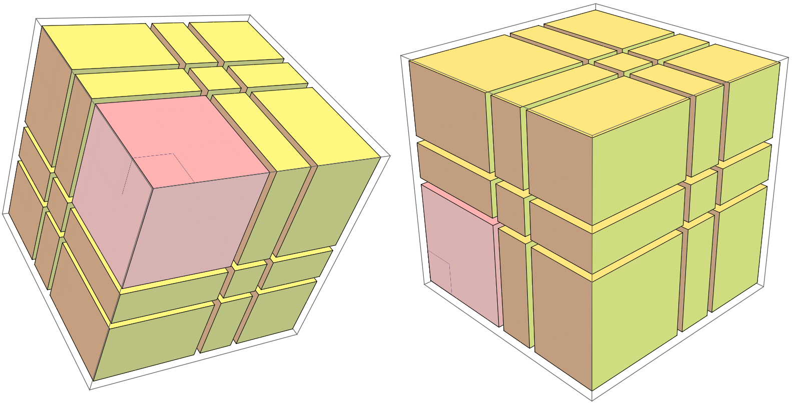
sizes : 1, 2, 3, 5

$\cdots \mapsto$

i	m	i	i	**j**	i	m	i
h	d	h	c	d	h	d	h
i	m	i	**l**	m	i	m	i
h	d	c	h	d	h	d	**e**
i	m	k	i	m	**g**	b	i
i	**j**	i	f	b	i	m	i
c	d	**e**	h	d	h	d	h
a	b	i	i	m	i	m	i

$\mapsto \cdots$

size : $8, \ldots$



Initial blocks of some 3-dimensional shape-symmetric picture

Maes' thesis p. 107.

Theorem (Maes 1999)

- Determining whether or not a map $\mu\colon \mathcal{B}_d(A) \to \mathcal{B}_d(A)$ is a d-dimensional morphism is a decidable problem.
- If μ is prolongable on a letter a, then it is decidable whether or not the fixed point $\mu^\omega(a)$ is shape-symmetric.

Theorem (Duchêne, Fraenkel, Nowakowski, R.)

The image by μ_W of the fixed point $\varphi_W^\omega(a)$ gives exactly the $\mathcal{P}$-positions of Wythoff's game.

Sketch of the proof of Maes's results

Cobham, Dumont–Thomas, Maes, Shallit, ...

$$\boxed{\text{Morphism} \leftrightarrow \text{Automata}}$$

Links with non-standard numeration systems: J. Shallit (1988), J.-P. Allouche, E. Cateland, et al. (1997), J.-P. Allouche, K. Scheicher, R. Tichy (2000), Marsault–Sakarovitch, M. R., ...

General theorem "morphic ⇒ automatic"

Let A be an ordered alphabet. Let $\mathbf{w} \in A^{\mathbb{N}}$ be an infinite word, fixed point $f^\omega(a)$ of a morphism $f : A^* \to A^*$.

- associate with f a DFA $\mathcal{M}$ over the alphabet $\{0, \ldots, \max |f(b)| - 1\}$;
- A is the set of states;
- the initial state is a, all states are final;
- if $f(b) = c_0 \cdots c_m$, then $b \xrightarrow{j} c_j$, $j \leq m$;
- consider the language L accepted by $\mathcal{M}$ except words starting with 0;
- genealogically order L: $L = \{w_0 < w_1 < w_2 < \cdots\}$.

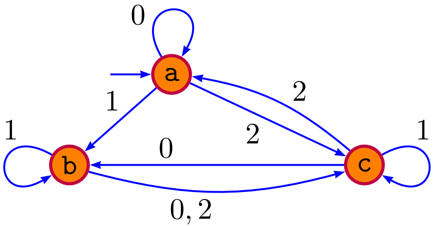
The nth symbol of $\mathbf{w}$, $n \geq 0$, is

$$\boxed{\mathcal{M} \cdot w_n}.$$

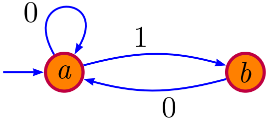
Examples (first, in 1D):

- Take your favorite k-uniform morphism, the associated regular language is $\{\varepsilon\} \cup \{1, \dots, k-1\}\{0, \dots, k-1\}^*$

$$f : \begin{cases} \mathtt{a} & \mapsto & \mathtt{abc} \\ \mathtt{b} & \mapsto & \mathtt{cbc} \\ \mathtt{c} & \mapsto & \mathtt{bca} \end{cases}$$



- Take the Fibonacci morphism $a \mapsto ab$, $b \mapsto a$, the associated regular language is $\{\varepsilon\} \cup 1\{0, 01\}^*$



We can do the same in a multidimensional setting.

- There are $d \geq 2$ associated regular languages (details missing, idea on the next slide).

Assume that the images of letters have shape (s_1, s_2), $s_i \leq 2$. Associate with φ an automaton with input alphabet:

$$\left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$$

$$\varphi(r) = \begin{array}{|c|c|}\hline u & v \\ \hline s & t \\ \hline \end{array}, \quad \begin{array}{|c|c|}\hline s & t \\ \hline \end{array}, \quad \begin{array}{|c|}\hline u \\ \hline s \\ \hline \end{array} \quad \text{or} \quad \begin{array}{|c|}\hline s \\ \hline \end{array}$$

we have transitions like

$$r \xrightarrow{\begin{pmatrix} 0 \\ 0 \end{pmatrix}} s, \quad r \xrightarrow{\begin{pmatrix} 1 \\ 0 \end{pmatrix}} t, \quad r \xrightarrow{\begin{pmatrix} 0 \\ 1 \end{pmatrix}} u, \quad r \xrightarrow{\begin{pmatrix} 1 \\ 1 \end{pmatrix}} v.$$

Associated languages — example of product of substitutions

$$f : \begin{cases} a & \mapsto & abc \\ b & \mapsto & cbc \\ c & \mapsto & bca \end{cases} \qquad g : \begin{cases} 0 & \mapsto & 01 \\ 1 & \mapsto & 0 \end{cases}$$

$f \times g$:

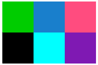
$$(a,0) \mapsto \begin{array}{|c|c|c|} \hline (a,1) & (b,1) & (c,1) \\ \hline (a,0) & (b,0) & (c,0) \\ \hline \end{array} \qquad (a,1) \mapsto \begin{array}{|c|c|c|} \hline (a,0) & (b,0) & (c,0) \\ \hline \end{array}$$

$$(b,0) \mapsto \begin{array}{|c|c|c|} \hline (c,1) & (b,1) & (c,1) \\ \hline (c,0) & (b,0) & (c,0) \\ \hline \end{array} \qquad (b,1) \mapsto \begin{array}{|c|c|c|} \hline (c,0) & (b,0) & (c,0) \\ \hline \end{array}$$

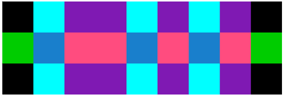
$$(c,0) \mapsto \begin{array}{|c|c|c|} \hline (b,1) & (c,1) & (a,1) \\ \hline (b,0) & (c,0) & (a,0) \\ \hline \end{array} \qquad (c,1) \mapsto \begin{array}{|c|c|c|} \hline (b,0) & (c,0) & (a,0) \\ \hline \end{array}$$

$\{\varepsilon\} \cup \{1,2\}\{0,1,2\}^*$ and $\{\varepsilon\} \cup 1\{0,01\}^*$

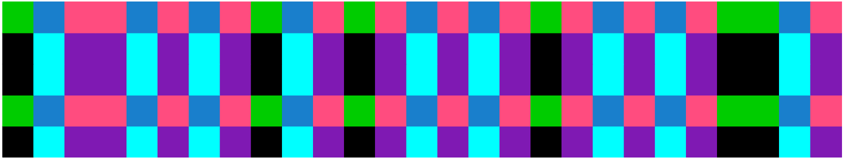
The growth is derived from these languages

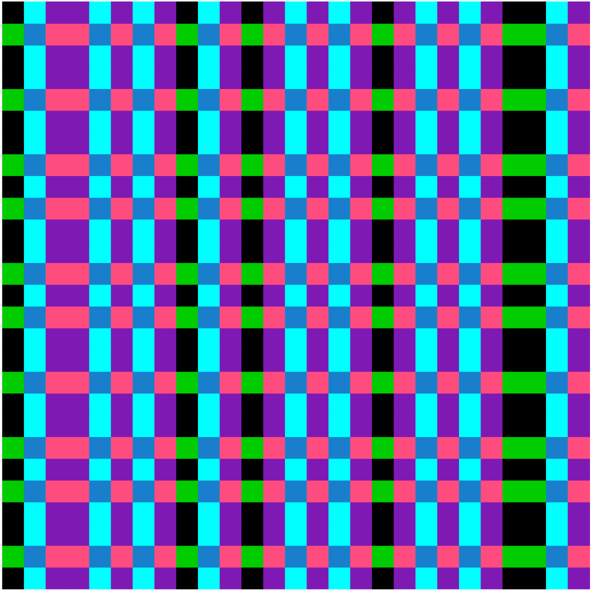


The growth is derived from these languages



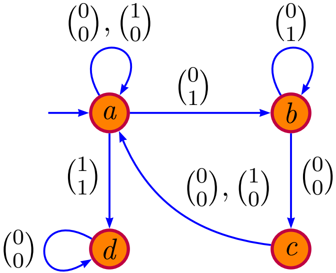
The growth is derived from these languages





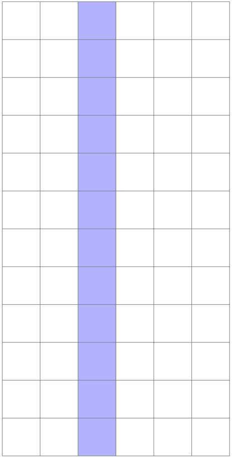
Can this map be extended to a morphism?

$$\gamma : \; a \mapsto \begin{array}{|c|c|}\hline b & d \\ \hline a & a \\ \hline \end{array}\,, \quad b \mapsto \begin{array}{|c|}\hline b \\ \hline c \\ \hline \end{array}\,, \quad c \mapsto \begin{array}{|c|c|}\hline a & a \\ \hline \end{array}\,, \quad d \mapsto \begin{array}{|c|}\hline d \\ \hline \end{array}\,.$$



Recall the condition $(\star)$:

$$\forall i \in \{1, \ldots, d\}, \forall k < |x|_i, \forall a, b \in \mathsf{Alph}(x_{|i,k}) : |\gamma(a)|_i = |\gamma(b)|_i.$$



$$\begin{pmatrix} 10 \\ 00 \end{pmatrix}, \begin{pmatrix} 10 \\ 01 \end{pmatrix}, \begin{pmatrix} 10 \\ 10 \end{pmatrix}, \begin{pmatrix} 10 \\ 11 \end{pmatrix}$$

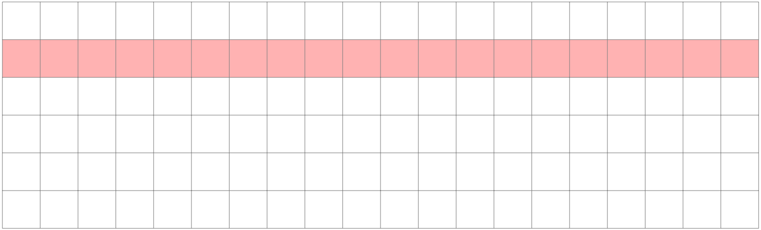
$$\begin{pmatrix} 010 \\ 100 \end{pmatrix}, \begin{pmatrix} 010 \\ 101 \end{pmatrix}, \ldots$$

$$\begin{pmatrix} 0^{|w|-2}10 \\ w \end{pmatrix}$$

The image by γ of all these elements should have the same number of columns.

Recall the condition $(\star)$:

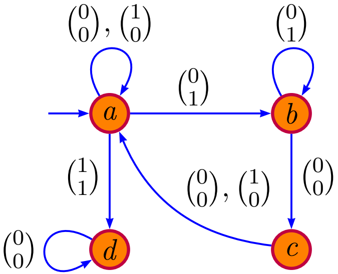
$$\forall i \in \{1, \dots, d\}, \forall k < |x|_i, \forall a, b \in \mathsf{Alph}(x_{|i,k}) : |\gamma(a)|_i = |\gamma(b)|_i .$$



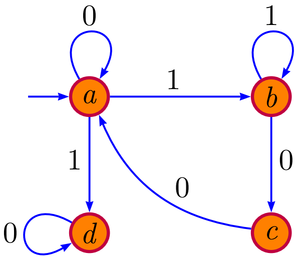
$$\binom{000}{100}, \binom{001}{100}, \binom{010}{100}, \binom{011}{100}, \binom{100}{100}, \dots$$

$$\binom{w}{0^{|w|-3}100}$$

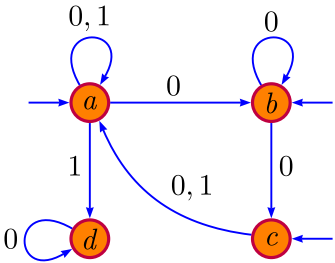
The image by γ of all these elements should have the same number of rows.



$\binom{0}{0}, \binom{1}{0}$
$\binom{0}{1}$
$\binom{0}{1}$
a
b
$\binom{1}{1}$
$\binom{0}{0}, \binom{1}{0}$
$\binom{0}{0}$
$\binom{0}{0}$
d
c



0
1
1
a
b
1
0
0
0
d
c



0, 1
0
0
a
b
1
0, 1
0
0
d
c

- Take the projections of the DFA $\mathcal{A}$
- We get 2 NFAs: $\mathcal{N}_1$ and $\mathcal{N}_2$
- The set of initial states is made of those reached by 0^*
- Determinize (Rabin–Scott's subset construction): $\mathcal{D}_1$ and $\mathcal{D}_2$

$Q = \{q_1, \ldots, q_r\}$ is a state of $\mathcal{D}_1$ reached when reading w,

IFF, in $\mathcal{N}_1$, there is a path from I_1 to q_j with label w, $\forall j$,

IFF, in $\mathcal{A}$, $\forall j$, there is a path from the initial state to q_j with a label of the form

$$\begin{pmatrix} 0 \cdots 0w \\ z_j \end{pmatrix}.$$

$\gamma(q_1), \ldots, \gamma(q_r)$ must have the same number of columns

- Take the projections of the DFA $\mathcal{A}$
- We get 2 NFAs: $\mathcal{N}_1$ and $\mathcal{N}_2$
- The set of initial states is made of those reached by 0^*
- Determinize (Rabin–Scott's subset construction): $\mathcal{D}_1$ and $\mathcal{D}_2$

$Q = \{q_1, \ldots, q_r\}$ is a state of $\mathcal{D}_2$ reached when reading w,

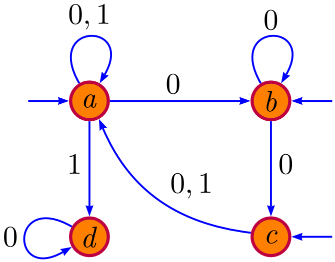
IFF, in $\mathcal{N}_2$, there is a path from I_1 to q_j with label w, $\forall j$,

IFF, in $\mathcal{A}$, $\forall j$, there is a path from the initial state to q_j with a label of the form

$$\begin{pmatrix} z_j \\ 0 \cdots 0w \end{pmatrix}.$$

$\gamma(q_1), \ldots, \gamma(q_r)$ must have the same number of rows

$$\gamma:\quad a \mapsto \begin{array}{|c|c|}\hline b & d \\ \hline a & a \\ \hline\end{array}\,, \quad b \mapsto \begin{array}{|c|}\hline b \\ \hline c \\ \hline\end{array}\,, \quad c \mapsto \begin{array}{|c|c|}\hline a & a \\ \hline\end{array}\,, \quad d \mapsto \begin{array}{|c|}\hline d \\ \hline\end{array}\,.$$



	state of $\mathcal{D}_2$	$\lvert\gamma(\cdot)\rvert_2$
$\mathcal{D}_2 \cdot \varepsilon$	$\{a, b, c\}$	$2, 2, 1$
$\mathcal{D}_2 \cdot 1$	$\{a, d\}$	$2, 1$
$\mathcal{D}_2 \cdot 10$	$\{a, b, d\}$	$2, 2, 1$
$\mathcal{D}_2 \cdot 100$	$\{a, b, c, d\}$	$2, 2, 1, 1$

Theorem (Maes 1999)

- If μ is prolongable on a letter a, then it is decidable whether or not the fixed point $\mu^\omega(a)$ is shape-symmetric.

IFF the associated languages are the same.

Is there some time left?

Theorem (Duchêne, Fraenkel, Nowakowski, R.)

The image by μ_W of the fixed point $\varphi_W^\omega(a)$ gives exactly the $\mathcal{P}$-positions of Wythoff's game.

We associate with φ an automaton with input alphabet

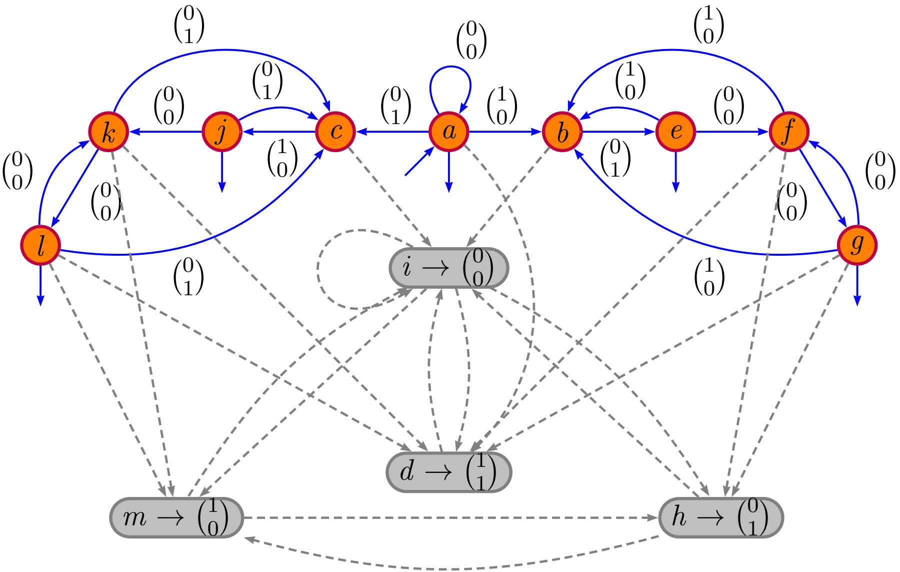
$$\left\{\binom{0}{0},\binom{1}{0},\binom{0}{1},\binom{1}{1}\right\}$$

$$\varphi(r)=\begin{array}{|c|c|}\hline u & v\\\hline s & t\\\hline\end{array}\,,\quad \begin{array}{|c|c|}\hline s & t\\\hline\end{array}\,,\quad \begin{array}{|c|}\hline u\\\hline s\\\hline\end{array}\quad \text{or}\quad \begin{array}{|c|}\hline s\\\hline\end{array}$$

we have transitions like

$$r\xrightarrow{\binom{0}{0}}s,\quad r\xrightarrow{\binom{1}{0}}t,\quad r\xrightarrow{\binom{0}{1}}u,\quad r\xrightarrow{\binom{1}{1}}v.$$

From morphism to automaton, we get



1) If all states are assumed to be final, this automaton accepts the words

$$\begin{pmatrix} u \\ v \end{pmatrix}$$

where $|u| = |v|$ and u, v are both valid F-representation (possibly padded with zeroes).

2) If we restrict to the "blue" part, this automaton accepts the words

$$\begin{pmatrix} 0w_1 \cdots w_\ell \\ w_1 \cdots w_\ell 0 \end{pmatrix} \text{ and } \begin{pmatrix} w_1 \cdots w_\ell 0 \\ 0w_1 \cdots w_\ell \end{pmatrix}$$

where $w_1 \cdots w_\ell$ is a valid F-representation.

3) Now, if the set of final states is $\{a, e, g, j, l\}$, we have the extra condition that $w_1 \cdots w_\ell$ ends with an even number of zeroes.

With Fraenkel's characterization of $\mathcal{P}$-positions, this concludes the proof. □

Theorem (A. S. Fraenkel, 1982)

(x, y), with $x < y$, is a $\mathcal{P}$-position of Wythoff's game iff $\mathrm{rep}_F(x)$ ends with an even number of zeroes and $\mathrm{rep}_F(y) = \mathrm{rep}_F(x)0$.

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