2-uniform words: cycle graphs, and an algorithm to verify specific word-representations of graphs

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Introduction

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- An alternation between the letters i and j in the word corresponds to an edge (i, j) in the graph.
- $G(w)$ is the graph generated by considering all possible alternations in w.
- If $G(w) = G$ for some word w, then w is said to be a word-representation or word-representant of the graph G.

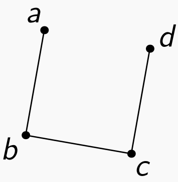

Figure: $G(w)$ for $w = bcabadc$.

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Theorem (Collins, Kitaev)

The number of n-vertex word-representable graphs is $2^{\frac{n^2}{3}+o(n^2)}$.

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The fundamental word

Define the word w_n, on the alphabet $\{1, 2 \ldots n\}$, by

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It is easy to see that $G(w_n) = C_n$ for every $n > 3$. Our claim is that from w_n, we can 'obtain' every word-representation of C_n. We look at one last definition before going on to the proof.

Circle Representations

Observation

If w' is the word obtained by a cyclic shift or a reflection of a 2-uniform word w, then $G(w') = G(w)$.

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Definition

- Represent a 2-uniform word w of length l on a circle, labelled by the letter occurring at each position from 1 to l, clockwise.
- We can imagine joining the two points where a specific letter repeats by a chord.
- Note that two letters alternate iff their corresponding chords intersect.

We call this the **circle representation** of w.

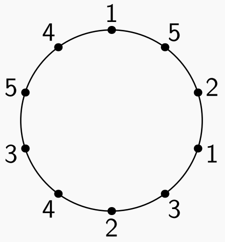

Figure: The circle representation of w_5, with chords not shown, and the start point as the topmost point. Each label represents the number at the position. We can start at any of the 10 positions, and read in either the clockwise or the anticlockwise direction, to get a total of $4 \times 5 = 20$ distinct words.

Main Theorem

Let w be any 2-uniform word that represents C_n labelled $1, 2, 3 ... n$ for $n > 3$, and let $w[i]$ be the letter at position i for all $1 \leq i \leq 2n$.

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Lemma

For every $r \in \{1, 2, 3 \ldots n\}$, the two sets of positions,

$$U_r = \{i : (w[i] - r) > 1\} \text{ and } L_r = \{i : (r - w[i]) > 1\},$$

if both are non-empty, lie entirely in one of the two segments defined by the chord corresponding to r. If exactly one is non-empty, then that set lies entirely in one of two segments.

The Main Theorem - Visual

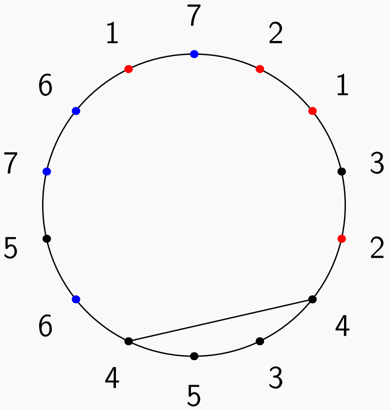
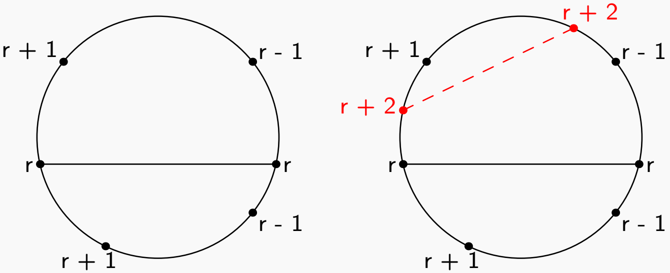


Figure: Circle representation of $w_7 = 17213243546576$, representing C_7. The chord corresponding to $r = 4$ has been drawn. The two sets of points, U_r and L_r have been coloured in blue and red, respectively. Black points belong to neither of the two sets.

The Proof Idea - Visual



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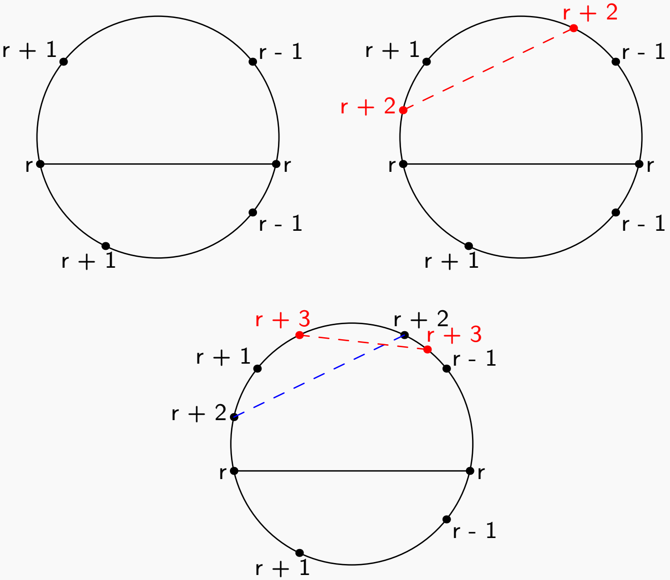

The Graphcheck Algorithm - Preliminaries

Following on from the circle representation idea, we aim to obtain an algorithm that will help us check if a given word 2-uniform word w is a word-representation of a given graph G.

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For this, we utilize a data structure that handles **Dynamic Range Sum Queries** efficiently - the Fenwick Tree.

The Fenwick Tree (Tarjan et al.)

- Used to calculate prefix sums of an array - say, of length n.
- $O(n)$ additional space.
- $O(n \log n)$ time initialization for an arbitrary array. (Here, however, this will not be necessary.)
- $O(\log n)$ time for a point update.
- $O(\log n)$ for an arbitrary prefix sum.

The Graphcheck Algorithm

Algorithm Details

- **Input**: 2-uniform word w on V, and graph $G = (V, E)$.
- **Result**: Returns **true** if $G(w) = G$, and **false** otherwise.

Initialization

- Initialize FenwickTree with 0 in all positions with total length $w.length()$.
- Initialize array of positions $pos[]$ to (NULL, NULL) for all letters in w.
- edgecount $= 0$

```
for k = 0 to w.length() − 1 do
    if pos[w[k]].first = NULL then
        pos[w[k]].first = k
    else
        pos[w[k]].second = k

        i = pos[w[k]].first
        j = pos[w[k]].second

        // add the number of unmarked nodes in w[i...j]
        edgecount +=
         j − i − FenwickTree.rangesum(i + 1, j − 1) − 1

        // mark the positions i and j
        FenwickTree.update(i, 1)
        FenwickTree.update(j, 1)
    end
end
```

```
if edgecount ≠ |E| then
    return false
else
    for edge (u, v) in E do
        if u and v do not alternate then
            return false ;            // only a O(1) comparison
        end
    end
    return true
end
```

Algorithm 1: GraphCheck

References and Credits

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Thank you!